An Essay by the Uniquely Wise 'Abel Fath Omar Bin Al-Khayyam on Algebra and Equations

Algebra wa Al-Muqabala

Omar Al-Khayyam

Translated by Professor Roshdi Khalil

Reviewed by Professor Waleed Deeb

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ON ALGEBRA AND EQUATIONS
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ALGEBRA WA AL-MUQABALA: An Essay by the Uniquely Wise
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In the Name of God, the Beneficent, the Merciful

FOREWORD

The interrelationship and interaction of human cultures and civilizations has made the contributions of each the common heritage of men in all ages and all places. Early Muslim scholars were able to communicate with their Western counterparts through contacts made during the Crusades; at Muslim universities and centres of learning in Muslim Spain (al-Andalus, or Andalusia) and Sicily to which many European students went for education; and at the universities and centres of learning in Europe itself (such as Salerno, Padua, Montpellier, Paris, and Oxford), where Islamic works were taught in Latin translations. Among the Muslim scholars well-known in the centres of learning throughout the world were al-Rāzi (Rhazes), Ibn Sīnā (Avicenna), Ibn Rushd (Averroes), al-Khwārizmī and Ibn Khaldūn. Muslim scholars such as these and others produced original works in many fields. Many of them possessed encyclopaedic knowledge and distinguished themselves in many disparate fields of knowledge.

The Center for Muslim Contribution to Civilization was established in order to acquaint non-Muslims with the contributions Islam has made to human civilization as a whole. The Great Books of Islamic Civilization Project attempts to cover the first 800 years of Islam, or what may be called Islam’s Classical Period. This project aims at making available in English and other European languages a wide selection of works representative of Islamic civilization in all its diversity. It is made up of translations of original Arabic works that were produced in the formative centuries of Islam, and is meant to serve the needs of a potentially large readership. Not only the specialist and scholar, but the non-specialist with an interest in Islam and its cultural heritage will be able to benefit from the series. Together, the works should serve as a rich source for the study of the early periods of Islamic thought.

In selecting the books for the series, the Center took into account all major areas of Islamic intellectual pursuit that could be represented. Thus the series includes works not only on better-known subjects such as law, theology, jurisprudence, history and politics, but also on subjects such as literature, medicine, astronomy, optics and geography. The specific criteria used to select individual books were these: that a book should give a faithful and comprehensive account of its field; and that it should be an authoritative source. The reader thus has at his disposal virtually a whole library of informative and enlightening works.

Each book in the series has been translated by a qualified scholar and reviewed by another expert. While the style of one translation will naturally differ from another as do the styles of the authors, the translators have endeavoured, to
the extent it was possible, to make the works accessible to the common reader. As a rule, the use of footnotes has been kept to a minimum, though a more extensive use of them was necessitated in some cases.

This series is presented in the hope that it will contribute to a greater understanding in the West of the cultural and intellectual heritage of Islam and will therefore provide an important means towards greater understanding of today’s world.

May God Help Us!

Muhammad bin Hamad Al-Thani
Chairman of the Board of Trustees
ABOUT THIS SERIES

This series of Arabic works, made available in English translation, represents an outstanding selection of important Islamic studies in a variety of fields of knowledge. The works selected for inclusion in this series meet specific criteria. They are recognized by Muslim scholars as being early and important in their fields, as works whose importance is broadly recognized by international scholars, and as having had a genuinely significant impact on the development of human culture.

Readers will therefore see that this series includes a variety of works in the purely Islamic sciences, such as Qur\textsuperscript{a}n, \textit{hadith}, theology, prophetic traditions (\textit{sunna}), and jurisprudence (\textit{fiqh}). Also represented will be books by Muslim scientists on medicine, astronomy, geography, physics, chemistry, horticulture, and other fields.

The work of translating these texts has been entrusted to a group of professors in the Islamic and Western worlds who are recognized authorities in their fields. It has been deemed appropriate, in order to ensure accuracy and fluency, that two persons, one with Arabic as his mother tongue and another with English as his mother tongue, should participate together in the translation and revision of each text.

This series is distinguished from other similar intercultural projects by its distinctive objectives and methodology. These works will fill a genuine gap in the library of human thought. They will prove extremely useful to all those with an interest in Islamic culture, its interaction with Western thought, and its impact on culture throughout the world. They will, it is hoped, fulfil an important rôle in enhancing world understanding at a time when there is such evident and urgent need for the development of peaceful coexistence.

This series is published by the Center for Muslim Contribution to Civilization, which serves as a research centre under the patronage of H.H. Sheikh Muhammad bin Hamad al-Thani, the former Minister of Education of Qatar who also chairs the Board of Trustees. The Board is comprised of a group of prominent scholars. These include His Eminence Sheikh Al-Azhar, Arab Republic of Egypt, and Dr Yousef al-Qaradhawi, Director of the Sira and Sunna Research Center. At its inception the Center was directed by the late Dr Muhammad Ibrahim Kazim, former Rector of Qatar University, who established its initial objectives.

The Center was until recently directed by Dr Kamal Naji, the Foreign Cultural Relations Advisor of the Ministry of Education of Qatar. He was assisted by a Board comprising a number of academicians of Qatar University, in addition to a consultative committee chaired by Dr Ezzeddin Ibrahim, former Rector of the University of the United Arab Emirates. A further committee
acting on behalf of the Center has been the prominent university professors who act under the chairmanship of Dr. Raji Rammuny, Professor of Arabic at the University of Michigan. This committee is charged with making known, in Europe and in America, the books selected for translation, and in selecting and enlisting properly qualified university professors, orientalists and students of Islamic studies to undertake the work of translation and revision, as well as overseeing the publication process.
CENTER FOR MUSLIM CONTRIBUTION TO CIVILIZATION

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Following are the names of the late prominent Muslim figures who (may Allah have mercy upon them) passed away after they had taken vital roles in the preliminary discussions of the Center’s goals, work plan and activities. They are:

1. Dr Kamal Naji, former General Supervisor, Center for Muslim Contribution to Civilization, Qatar (7 October 1997).
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CENTER’S ADVISORS

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Professor Raji Mahmoud Rammuny
Director of the Center’s Translation Committee in the U.S.A.
Professor of Arabic Studies, Department of Near Eastern Studies, University of Michigan, U.S.A.
When I accepted the challenge of translating Omar Al-Khayyam’s book from Arabic to English, I told myself this would be an easy job to do. After all, I consider myself to be good at both mathematics and the Arabic language. The shock came when I discovered that first I had to translate from the Arabic of Omar Al-Khayyam to my Arabic. That was the difficult part of my job. Another shock was to discover my ignorance of the scientific achievements of great scientists such as Omar Al-Khayyam. Going through the book, reading theorems, proofs and remarks, I realized that I was reading the work of a great mathematician. The preciseness of the statements and the accuracy of the proofs made me think that I was reading an article in a high-ranking recent mathematical journal.

Omar Al-Khayyam was born in the middle of the eleventh century in the city of Nishapur. He was a poet and a mathematician. I read some of his poetry when I was an undergraduate student. At that time I did not know that he was a mathematician and had written books on mathematics.

Omar Al-Khayyam’s book mainly deals with equations of degree at most three:

\[ ax^3 + bx^2 + cx + d = 0 \]

including all cases where some of the integer coefficients \(a, b, c, d\) equal zero.

At the time of Omar Al-Khayyam, the two equations

\[ ax^3 + bx^2 + cx + d = 0 \]
\[ ax^3 + bx^2 + cx = d \]

were regarded as two different cases of equations of degree three.

Al-Khayyam’s book studies and presents the following:

1. Third degree equations that can be reduced to equations of degree two.
2. Third degree equations that consist of three terms.
3. Third degree equations that consist of four terms.
4. Equations that involve the reciprocal of the unknown (variable).
5. The problem of dividing a quarter of a circle into two parts with a given ratio.

Omar Al-Khayyam used geometry, especially conic sections, to prove his results.

I learned so much from the project of translating Omar Al-Khayyam’s book—*Algebra wa Al-Mugabala* (Algebra and equations). Just after I finished translating the book, I met Professor Roshdi Rashid, in Amman at a conference on the history
of Arab sciences. He drew my attention to the fact that there are actually two
different manuscripts of Al-Khayyam’s book. One copy is in Aleppo, the one that
I translated into English. The second copy is in France, the one he translated into
English. I thank Professor Rashid very much for his comments and the valuable
information he supplied me with.

I wish to thank Professor Raji Rammuny who suggested my name for the
project of translating Al-Khayyam’s book, thus offering me the chance to
explore part of the work of a great scholar of my culture.

I thank all those who helped me in my task in one way or another, in particular
my students S. Awamleh and A. Khawalda, who helped me in drawing the graphs
for the book.
One of the educational notions needed in the branch of philosophy known as mathematics is the art of algebra and equations, invented to determine unknown numbers and areas. It involves problems that reflect difficult propositions; most people studying such problems have been unable to solve them. In the case of the ancient ones [researchers], none of their work has reached us, either because they did not solve these problems despite trying, or because they did not need to solve them, or simply because their work was lost.

As for the modern ones, Mahani tried to analyze algebraically the proposition used by Archimedes as a postulate in proposition four of the second article of his algebra book on spheres and cylinders. In his analysis, he discovered equations involving squares and cubes of numbers that he could not solve, though he thought deeply about them. So, he concluded that such equations are impossible to solve. No one could solve such equations, until the genius “Abu Jafar Al-Khazin” solved them using conic sections.

Later on, a group of geometers needed certain classes of these equations. Some of them (the geometers) solved certain types of these problems. But none of them did any sound work concerning the classification and sub-classification of such problems or the proof of any, except for two classes that I will mention later in this book. And I was, and remain, very keen to classify the problems and indicate (by proof) those that can be solved and those that are impossible to solve, since I knew the very need of it in solving other (open) problems.

I was unable to devote myself completely to achieving this worthy task, or to pursuing my ideas generally, because the demands of my daily life were a necessary diversion.

We have been afflicted in our time by a lack of scientists, except for a particular group, few in number but many in the troubles that beset them, whose concern is to exploit any gleam of trouble-free time to achieve and articulate some branches of science.

Many of those who pretend to be wise men in our time defraud the truth with falsity and do not seek to move forward the frontiers of knowledge, preferring instead to use the little they know of science for low materialistic goals. And once they meet someone who sincerely wishes to acquire facts and who prefers truth, trying to reject falsity and avoid deception, they make fun of him and ridicule him. So may God be our helper and our comforter.

God gave me the opportunity of being with our unique great master, the head judge, the scholar Imam Abi Tahir, may God keep his high position, and
keep silent his enviers and enemies. Once I despaired to meet someone like him, perfect in every virtue, theoretical or practical, who can work deeply in science, verify others’ work and seeks the welfare of everyone of his kind; I was so delighted to see him. I achieved fame through his companionship. My affairs were glorified by his illuminations, and my support intensified through his grace. It was my opportunity to benefit from my new status.

So I started to summarize what I can verify of deep knowledge so as to be closer to the master (Abi Tahir). Since the priority is mathematics, I started to list the algebraic propositions.

I adhered to the guidance of God, and I entreated God to grant me success in verifying my scientific research and the important research of those before me, grasping the most trustworthy of God’s protection. For He is the one who answers our prayers and on whom we depend.

With the help of God, and with his gentle guidance, I say:

The study of algebra and equations is a scientific art. The ingredients are the absolute numbers and unknown measurable quantities, which are related to a known quantity. Each known thing is either a quantity or a unique relation that can be determined by careful examination.

By quantities we mean continuous quantities, and they are of four types: line, surface, solid, and time – as mentioned briefly in Categories, the book of Aristotle, and in detail in his other book, Metaphysics. Some (researchers) consider place to be a continuous quantity of the same type as surface. This is not the case, as one can verify. The truth is: space is a surface with conditions, whose verification is not part of our goal in this book. It is not usual to mention time as an object in algebraic problems. But if it were mentioned, it would be quite acceptable.

It is a practice of the algebraists (in their work) to call the unknown to be determined an object (variable), and the product of the object by itself a square (maal). The product of the object by its square is called a cube, and the product of the square by the square: square-square (maal-maal); the product of cube by square: square-cube, and the product of cube by cube: cube-cube, and so on.

It is known from the Elements, the book of Euclid, that these ranked products are all proportional in the sense that the ratio of one to the root is as the ratio of root to square, is as the ratio of squares to cubes. So the ratio of the numbers to roots is as the ratio of roots to squares, is as the ratio of squares to cubes, and so on.

It has to be clear that for anyone to be able to understand this essay, he has to be acquainted with the two books of Euclid (the Elements and the Data) and two chapters of the book of Apollonius on Cones. Whoever lacks knowledge of any of these three references, will not understand this essay. I have taken pains in trying not to refer to any article or book except those three books.

Algebraic solutions are achieved using equations. I mean, as is well known, by equating the ranks (powers) one with the other. If an algebraist uses square-square in areas, then this would be figuration not fact, since it is impossible for square-square to exist in measurable quantities. What we get in measurable
quantities is first one dimension, which is the “root” or the “side” in relation to its square; then two dimensions, which represent the surface and the (algebraic) square representing the squared surface (rectangle); and finally three dimensions, which represent the solid. The cube in quantities is the solid bounded by six squares (parallelepiped), and since there is no other dimension, the square of the square does not fall under measurable quantities let alone higher powers.

If it is said that the square of the square is among measurable quantities, this is said with reference to its reciprocal value in problems of measurement and not because it in itself is measurable. There is a difference between the two cases.

The square of the square is, therefore, neither essentially nor accidentally a measurable quantity and is not as even and odd numbers, which are accidentally included in measurable quantities, depending on the way in which they represent continuous measurable quantities as discontinuous.

Of the four (geometrical) equations that involve the absolute numbers, sides, squares and cubes, the books of the algebraists contain only three of these equations involving numbers, sides and squares. But we will give the methods by which one can determine the unknown using equations that involve the four measurable quantities that we mentioned, I mean: the number, the object, the square and the cube. Whatever can be proved using the properties of the circle, I mean using the two books of Euclid, the Elements and the Data, will be given simpler proofs. But those that cannot be proved except by using conic sections will be proved using two articles on conics.

As for the proofs of these types (of problems): if the problem is concerned only with the absolute number, then we cannot (in general) supply the proofs (and no one who works in this industry can). Hopefully, someone who comes after us will be able to (supply the proofs). As for the first three ranks: number, object and square, we will supply the proofs, and I may refer to numerical proofs of problems that can be proved using the book of Euclid (the Elements).

You have to know that the geometrical proofs of these problems do not dispense the numerical proofs if the topic is the number not the measurable quantities. Can you not see that Euclid proved certain equations to find the required rational measurable quantities in chapter five of his book (the Elements), and then resumed his proof of such problems, in chapter seven of his book, to determine these required ratios, if the topic is some number.

Equations involving these four types are either simple or multi-term equations. The simple equations are of six types:

(1) Number equals root
(2) Number equals square
(3) Number equals cube
(4) Roots equal square
(5) Squares equal cube
(6) Roots equal cube.
Three of these six equations are mentioned in the books of the algebraists. They (the algebraists) said: the ratio of the object to the square is as the ratio of the square to the cube. So equating the square to the cube is as equating the object to the square. Further, the ratio of the number to the square is as the ratio of the root to the cube. So it follows that the equation of the number and the square is as the equation of the root and the cube. They (the algebraists) did not prove that using geometry.

As for the number that equals the (volume) cube, there is no way to determine it (the number) except through mathematical induction. If a geometrical method is to be used to determine the number, then it can only be done through conic sections.

As for multi-term equations, they are of two classes: three-term equations and four-term equations. The three-term equations are of twelve types. The first three of them are:

1. Square and a root equal a number
2. Square and a number equal root
3. Root and a number equal square.

These three have been mentioned in the algebraists’ books, including their proofs using geometry, not using numbers. The second three (of the three-term equations) are:

1. Cube and square equal root
2. Cube and root equal square
3. Cube equals root and square.

The algebraists said that these three equations are equivalent to the first three, each to the corresponding one. I mean: cube and root equal square is equivalent to square and number equal root. And the other two are the same. They (the algebraists) did not prove them if the topic (unknown) of the problem is area. But they did solve them if the unknown is number, as is clear in the book of Elements. I will prove the geometrical ones.

The other six types of the twelve types are:

1. Cube and root equal number
2. Cube and number equal root
3. Number and root equal cube
4. Cube and square equal number
5. Cube and number equal square
6. Number and square equal cube.

These six types were never mentioned in their (the algebraists’) books, except for one, where the proof was not complete. I will prove all of these types using geometry, not numeric. The proofs of these six types can only be deduced through the properties of conic sections.
As for the four-term equations, they consist of two groups. The first group, where three terms equal one term, contains four types:

1) Cube, square and root equal number
2) Cube, square and number equal root
3) Cube, root and number equal square
4) Cube equals root, square and number.

The second group, in which two terms equal two terms, is of three types:

1) Cube and square equal root and number
2) Cube and root equal square and number
3) Cube and number equal root and square.

These are the seven types of the four-term equations, none of which can be solved except through geometry.

One of those (algebraists) who lived before us needed one type of one part of these equations, as I will mention. The proof of these types cannot be produced except through conic sections. We will prove the twenty-five types of these equations, one by one, asking help from God, for those who sincerely depend on Him will get help and guidance.

First type of simple equations: Root equals a number \((ax = b)\)

The root is necessarily known; this applies to numbers and areas.

Second type: Number equals a square \((x^2 = b)\)

So the square is known, being equal to the known number. There is no way to find the root except by trial. Those who know that the root of twenty-five is five know that by induction, not through a deduced formula, and one need pay no attention to people who differ in this matter. People of India have ways of finding the side of a square (knowing the area) and of a cube (knowing the volume) and these methods are based on simple induction, that depends on knowing the squares of the nine (numbers). I mean the square of one, two, three, and their products (that is to say the product of two by three and so on). We have written an article in which we prove the validity of these methods and show how it leads to the required results. We enriched its types, I mean by finding the sides of square-square, square-cube, and cube-cube, and so on. We were the first to do that. These proofs are numerical ones and are based on the numerical part of the book of *Elements*.

The following is the proof of the second type using geometry:

Draw the line \(ab\) whose length equals the given number and a line \(ac\) (of unit length) perpendicular to \(ab\) and then complete the surface (rectangle) \(ad\). The area of the surface \(ad\) is equal to the given number. We construct a square whose area equals the area of \(ad\); call it \(h\), as shown by Euclid, proposition zd, article b
of his book. The (area of) square $h$ equals the given number, which is known. Its side is also known. Examine the proof given by Euclid. This is the required result.

Whenever we say (in this essay) a number equals a surface (area of a rectangle), we mean there is a right-angle surface (rectangle) with one side equal to one unit and the other equals the given number, and each unit part of the area equals the other side, which we assumed to be one.

Third type: A number equals a cube
If the number is known, the cube (volume) is known and the only way to find its side is by induction. The same applies to higher orders like square-square, square-cube, and cube-cube, as we mentioned earlier.

But, using geometry we construct the square $ad$ to be of unit side, I mean $ac$ and $bc$ both equal one. Then, we draw a perpendicular to the surface $ad$ at the point $b$, call it $bc$, of length equal to the given number, as Euclid showed in section eleven of his book. We complete the parallelepiped $abchzm$.

It is known then that the area of this surface equals the given number. Now, we construct a cube of the same volume as this parallelepiped. Such a construction cannot be done without using properties of conic sections. We delay this until we introduce the background for this study. When we mentioned that a number equals a parallelepiped we mean there is a parallelepiped (with right angles) with a unit square base and its height equals the given number.
Fourth type: Square equals five of its root \((x^2 = 5x)\)

So the number of the roots (five) equals the root of the square. The numerical proof is as follows: If the root is multiplied by itself you get the square. But if this root is multiplied by five you get the square. So it must be five.

The geometric proof is similar to the numerical one. Just construct a square whose area is equal to five times the length of its side.

Fifth type: Roots equal cube \((ax = x^3)\)

Numerically: clearly, this is equivalent to a number equals a square. As an example: four times a root equals a cube, which is equivalent to four equals a square.

But geometrically: we construct a cube \(abcdh\) whose volume equals four times the length of its side \(ab\). So multiplying the length of the side \(ab\) by four gives the volume of the cube. But, multiplying the length of \(ab\) by the area of the square \(ac\) gives the volume of the cube again. So, the area of the square of \(ac\) equals four.

![Image of a cube with labels: a, b, c, d, h, showing the volume calculation](image)

Sixth type: Squares equal a cube \((ax^3 = x^3)\)

This is equivalent to a number equals a root \((a = x)\). To prove this numerically: the ratio of a number to a root is the same as the ratio of a square to the cube, as shown in chapter eight of Euclid’s book, the *Elements*.

Geometrically: we construct a cube \(abcdh\) whose volume is equal to the number of the squares of its side; for example twice the square of its side.

![Image of a cube with labels: a, b, c, d, h, showing the volume calculation](image)
If we multiply the area of \( ac \) by two, we get the volume of the cube \( abcdh \). Also, if we multiply the area of \( ac \) by (the length of) \( bd \) we get the volume of the cube again. So (the length of) \( bd \) must be equal to two. This is what we wanted to show.

Whenever we say in this article ‘the squares of the cube’, we mean the squares of its sides.

Now, we have finished the simple equations. Let us discuss the first three of the twelve types of three-term equations.

**First type: Square plus ten times its root equals thirty-nine \((x^2 + 10x = 39)\)**

Multiply half the number of the roots by itself. Add the result to the number of the roots. Then, subtract from the root of the outcome half the number of the roots. What is left is the root of the square.

The number must satisfy two conditions. The first is: the number of the roots must be even in order to have a half. The second is: the square of half the number of the roots plus the given number is a complete square. Otherwise, it is impossible to solve the problem numerically.

However, using geometry, none of the cases is impossible to solve. The numerical solution is much simpler if one visualizes the geometric solution.

The geometric solution is as follows:

We let the area of the square \( ac \) plus ten times its root equal thirty-nine. But ten times the root equals the area of the rectangle \( ch \), so the side \( dh \) equals ten.

We bisect \( dh \) at \( z \). Since \( dh \) was divided into two equal parts at \( z \), and extended to include \( ad \), then the product of \( ha \) and \( ad \) (this product equals the area of \( bh \)) plus the square of \( dz \), equals the square of \( za \).

But, the square of \( dz \), which equals half the number of the roots, is known. And the area of \( bh \), which is the given number, is also known. So the square of \( za \) is known, and so is the line \( za \). If we subtract \( zd \) from \( za \) we are left with \( ad \), and so \( ad \) is known.

There is another proof:

We construct the square \( abcd \), and extend \( ba \) to \( m \) so that \( ma \) equals one fourth of the number of the roots, which is two and a half. We extend \( da \) to \( z \) so
that \(za\) equals one fourth of the number of the roots. We construct similar lines from all corners of the square \(abcd\).

We complete the rectangle \(mt\), which is a square because \(zm\) is a square, \(ac\) is a square and \(ct\) is a square as shown in chapter six of the book of *Elements*.

Each of the four squares at the corners of the big square (\(abcd\)) has area equal to the square of two and a half. So the sum of all (areas) squares is twenty-five, which equals the square of half the number of the roots. Further, the (area of) rectangle \(zh\) equals two and a half of the roots (sides) of the square of \(ac\), because \(za\) equals two and a half, so the (area of the) four rectangles equals ten roots (side) of the square of \(ac\). But the square of \(ac\) and ten of its roots was assumed to equal thirty-nine. Hence the square of \(mt\) is sixty-four. Take its root and subtract five. What is left is \(ab\).

Now, if the (length) line \(ab\) is assumed to equal ten, and we need to have a square for which the product of the length of its side by the length of \(ab\) equals the given number, then we let the given number equal the area of a rectangle \(h\).

We construct, on the line \(ab\), a rectangle equal (in area) to the rectangle \(h\), then we construct a square on the side of the new rectangle, say \(bd\), as shown by Euclid in chapter six of his book, the *Elements*. So the added square \(ad\) and its side \(ac\) are known, as shown in *Data*.
Second type: Square plus a number equals roots ($x^2 + a = bx$)

In this case, the number should not exceed the square of half the number of the roots; otherwise the problem has no solution.

In case the number is equal to the square of half the number of the roots, then half the number of the roots is the root of the square. In case it is less than the square of half the number of the roots, then subtract the number from the square of half the number of the roots, and take the root of the result. Add (or subtract from) the outcome to half the number of the roots. What we get after adding or subtracting is the root of the square.

The numerical proof can be visualized, once the geometric proof is understood:

We construct the square $abcd$, and the rectangle $hd$ on the side $ad$ so that, for example, the area of $hc$ is ten times the side of the square $ac$. This implies $hb$ equals ten. In the first case, $ab$ equals half $bh$, and in the second case, greater than its half, and in the third, less than its half. In the first case, $ab$ will be equal to five; while in the second and the third cases, we bisect $hb$ at $z$.

So, the line $hb$ is divided into two halves at $z$ and into unequal parts at $a$. So the product of $ha$ and $ab$ added to the square of $za$ equals the square of $zb$, as shown in the second chapter of the book of Elements.

But the product of $ha$ and $ab$ is the given number, which is known. If this product is subtracted from the square of $zb$, which equals half the number of the roots, we are left with the square of $za$, which is known. In the third case, subtract the product from $zb$. In the second case add to the product $az$. The outcome will be $ab$, which is the required result. If you wish, you can prove it in other ways, but we do not want to present them to avoid lengthy treatment.

But if it is assumed that the length of $ab$ is ten, say, and a segment is to be cut out from it ($ab$), so that the product of $ab$ times (the length of) that line equals
the square of that line plus the area of another rectangle, not larger than the square of half \( ab \), I mean the given number which is the area of the rectangle \( h \), and we want to cut from \( ab \) a line whose square plus the rectangle \( h \), equals the product of \( ab \) times (the length of) that line, we add to the given line \( ab \) a rectangle, say \( az \), whose area equals the known area of \( h \) minus the area of a square, say \( cd \), which is possible because the area of \( h \) is not larger than the square of half \( ab \), as shown by Euclid in chapter six of his book (the *Elements*). So the side of \( cb \) is known from the given data, and that is the required result.

It seems that this type has many kinds, some of them impossible to solve. You can find the conditions on the number, which make the problem solvable, based on what we have shown in our discussion of the first type.

Third type: Number and a root equal a square \((a + bx = x^2)\)

Add the square of half the number of the roots to the number. Then take the root of this sum and add it to half the number of the roots. The result will be the root of the square.

**Proof:** The square \( abcm \) equals five times its root plus six \((x^2 = 5x + 6)\). We subtract the number, which is the rectangle \( ad \). We are left with the rectangle \( hc \), which equals the number of the roots, which is five.

So the length of \( hb \) equals five. We divide \( hb \) into two equal parts at \( z \). Hence, the line \( ab \) is divided in two equal parts at \( z \). Add to it \( ha \) by extension.

So the product of \( ba \) and \( ah \) (which is equal to the area of the known rectangle \( ad \)) plus the known square of \( hz \), equals the square of \( za \). Consequently, since the square of \( za \), \( za \) itself, and \( zb \) are all known, we conclude that \( ab \) is known. There are other methods to prove this. We accept this one.
But if it is assumed that $bh$ equals the number of the roots, and it is required to find a square for which the square plus its side equals the number of the sides plus the given number, then $abcd$ is the required square.

To see that, assume that the given number is the area of the rectangle $t$, and the square that is equal to it is $m$. We construct an equal square (to $m$), say $z$. We construct $kc$ to equal the side of $z$; then we complete the square $abcd$. The square $abcd$ is the required square.

We have shown that in this third type, no case is impossible to solve. This is true also for the first type. For the second type there are cases with no solution, and it involves cases that do not occur in the first and the third type.

**Equations of the third degree that can be reduced to second degree equations**

The proof that the second three equations are equivalent to the first three is as follows.

**First type: Cube and squares equal roots ($x^3 + ax^2 = bx$)**

We construct a cube $abcdh$. Extend $ab$ to $z$ so that $az$ equals the given number of the squares. Then complete the parallelepiped $azmtcd$ beside the cube $ah$, as usual. The parallelepiped $at$ equals the number of the squares. So the parallelepiped $bt$, which equals the cube plus the number of the given squares, equals the number of the given roots. Construct the surface (rectangle) $k$ to equal the number of the assumed roots. The root is the side of the cube, and it is $ad$. Hence, if the area of the rectangle $k$ is multiplied by the length of $ad$, the result will be equal to the number of the given roots. Now, if the (area) rectangle $mb$ is multiplied by $ad$, we get the cube plus the assumed number of squares. But these two quantities are equal.

I mean, the parallelepiped $bt$ and the parallelepiped constructed on the base $k$ with height $ad$ have the same volume. Since they have equal heights, they must have equal bases. The base of $mb$ is the square $cb$ with the square $ma$, which is the number of the roots of the assumed squares.
Hence, \( k \), which is the assumed number of the roots, equals a square plus the assumed number of the roots of the squares. That is what we wanted to show.

**Example:** A cube plus three squares equals ten roots is equivalent to a square plus three roots equals ten.

Second type: Cube and two roots equals three squares, which is equivalent to square plus two equals three roots \((x^3 + 2x = 3x^2 \leftrightarrow x' + 2 = 3x)\)

**Proof:** We construct the cube abcdh so that the cube plus two of its roots equals three squares. We then construct a square \( m \) to equal the square cb, and we set \( k \) equal to three. So the product of \( m \) with \( k \) equals three squares of the root of the cube ah. We construct on ac a rectangle that equals two, then we complete the solid (parallelepiped) azctd. Then azctd equals the number of the roots. But if the (length) line bz is multiplied by the square ac we get the (volume) parallelepiped bt. However, the parallelepiped at equals the number of roots. So the parallelepiped bt equals the cube plus the number of its sides. So the parallelepiped bt equals the number of the squares, and hence the length of bz is equal to three, as was shown earlier. The rectangle bl is a square plus two, so a square plus two equals three roots because the rectangle bl is the product of ab by three and that is what we wanted to show.
Third type: Cube equals a square and three roots \((x^3 = x^2 + 3x)\) which is equivalent to square equals a root and three \((x^2 = x + 3)\)

**Proof:** Construct the cube \(abcdh\) to equal a square and three of its sides. Then we remove from \(ab\) (which is the side of the cube) the segment \(az\), whose length equals the number of the squares which is equal to one. Complete the parallelepiped \(aztmc\) so its volume equals the assumed number of the squares. What remains is the parallelepiped \(zh\), which is equal to the number of the assumed sides. The ratio of the (volumes) of the two parallelepipeds equals the ratio of the base of \(zc\) to the base of \(zl\), as was shown in chapter eleven of the book of *Elements*, since their heights are equal. But the (area) of the rectangle \(zc\) is one root of the square \(cb\), and (area of) \(zl\) equals the number of the roots which is three. Thus, the square of \(cb\) equals a root and three, and that is what we wanted to show.

If our explanations of the proofs are not understood, then the proofs may appear not to be correct, though they were difficult to explain.

**Equations of degree three, composed of three terms**

After introducing these types, which we were able to prove using the properties of circles, mentioned in the book of Euclid, let us move now to types whose proofs can only be given using the properties of conic sections. There are fourteen of these types. One is a single-term equation, which is a cube equals a number, six consist of three-term equations and seven consist of four-term equations. Let us give an introduction based on the book of conic sections to serve as background for the student so that we will not need to refer to more than the three mentioned books, that is the two books of Euclid, the *Elements* and the *Data*, and two articles of the book of conic sections.

**Introduction 1 (Lemma 1):** We need to find two lines between two given lines so that the four are proportional.

Let the two given lines be \(ab, be\), making a right angle at \(b\). We construct the parabola \(bdh\), with vertex at \(b\), axis \(bc\) and its right side is \(bc\). So the conic...
section $bdh$ is known because its vertex and axis are known, and the magnitude of its right side is known. Further, the conic will be tangent to the line $ba$ because $b$ is a right angle and it is equal to the angle of order, as shown in article a of the book of conics.

Similarly, we construct another parabola, $bdz$, with vertex at $b$, axis $ab$, and its orthogonal side is $ab$. The conic $bdz$ is tangent to $bc$ as was shown by Apollonius in proposition no of article a. The two sections must intersect at the point $d$, say. The location of $d$ is known because the location of the two sections is known. We draw from $d$ two lines $dm$, $dt$ orthogonal to $ab$, $bc$ respectively. So their magnitudes are known, as shown in the *Data*. I claim: the four lines $ab$, $bm$, $bt$, $bc$ are proportional.

**Proof:** The square of $md$ equals the product of $bm$ and $bc$, because the line $dm$ is one of the lines of order of the section $bdh$. It follows that the ratio of $bc$ to $md$ (which is equal to the line $bt$) is as the ratio of $bt$ to $mb$. And the line $dt$ is one of the lines of order of the section $bdz$, so the square of $dt$ (which is equal to $bm$) equals the product of $ba$ and $bt$. Thus, the ratio of $bt$ to $bm$ is as the ratio of $bm$ to $ba$. Hence the four lines are successive and proportional, and the magnitude of the line $dm$ is known because it is drawn from a given point to a given line with a given angle. Further, the magnitude of $dt$ is known. So the two lines $bm$ and $bt$ are of known magnitude and they are middle in ratio between $ab$ and $bc$: I mean the ratio of $ab$ to $bm$ is the same as the ratio of $bm$ to $bt$ and is as the ratio of $bt$ to $bc$, and that is what we wanted to show.

**Introduction 2 (Lemma 2):** Given a square $abcd$, which is a base of a solid (parallelepiped) $abcdh$ with parallel surfaces and right angles; and given a square $nm$. We need to construct a parallelepiped such that its base is $nm$ and its volume is equal to the volume of $abcdh$.

We make the ratio of $ab$ to $nz$ equal to the ratio of $nz$ to $k$, and the ratio of $ab$ to $k$ equal to the ratio of $zt$ to $hd$. Then make $zt$ perpendicular to the square
nm at z, and complete the parallelepiped nztm. I claim that the volume of this parallelepiped equals the volume of the given one.

**Proof:** The ratio of the (area of the) square ac to the (area of the) square nm is the same as ab to k. Thus, the ratio of the (area of the) square ab to the (area of the) square nm is the same as the ratio of zt (which is the height of the parallelepiped ntm) to dh (which is the height of the parallelepiped bh). Hence, the two parallelepipeds have the same volume because their bases are equivalent to their heights as shown in article ya of the book of Elements.

Whenever we use the word “solid” we mean parallelepiped, by “a surface” we mean rectangle.

**Introduction 3 (Lemma 3):** The parallelepiped abcd is given, and its base ac is a square. We need to construct a parallelepiped whose base is a square and whose height equals the given ht, and whose volume equals that of abcd.

We make the ratio of ht to bd equal to the ratio of ab to k. Between ab and k, we construct a line which is middle in ratio between them, call it hz. We make hz perpendicular to ht, and complete the parallelepiped tz. Make mh perpendicular to the surface tz and equal to zh, and then complete the parallelepiped mhtz.

I claim that the parallelepiped tz, whose base is the square mz and its height is the given ht, is equal (in volume) to the given parallelepiped d.

**Proof:** The ratio of the square ac to the square mz is the same as the ratio of ab to k. So the ratio of the square ac to the square mz is the same as the ratio of ht to bd. The bases of the two parallelepipeds are equivalent to their heights, so they are equal, and that is what we wanted to show.

Now we discuss the third type of simple equations, which is a cube equals a number.

We set the number equal to the volume of a parallelepiped abcd whose base is ac, which is a unit square as we stated, and its height equals the given number. We want to construct a cube that is equal to it (in volume).
We construct two intermediate lines between the two lines \(ab\) and \(bd\), whose ratios are in between too; so their magnitudes are known as we stated. The two lines are named \(h\), \(z\), and we make \(mt\) equal to the line \(h\) on which we construct the cube \(tmkl\). Then, the volume of \(tmkl\) is known and the length of its side is known. I claim that its volume equals that of the parallelepiped \(d\).

**Proof:** The ratio of the square of \(ac\) to the square of \(tk\) is the same as twice the ratio of \(ab\) to \(mk\). Further, twice the ratio of \(ab\) to \(mk\) is the same as the ratio of \(ab\) to \(z\) (the first to the third of the four lines), which is the ratio of \(mk\) the second line to the fourth \(bd\).

So the bases of the cube \(l\) and the parallelepiped \(d\) are equivalent to their heights, so they have the same volume. That is what we wanted to show.

From this point on, we shall discuss the remaining six types of equations of three terms each.
First type: Cube plus sides (roots) equals a number \((x^3 + ax = b)\)

We set \(ab\) to be a side of a square whose length equals the given number of the roots. Then we construct a parallelepiped with a square base whose side is \(ab\) and its height is \(bc\), which we assume to equal the given number. The construction is similar to what we have done before. We make \(bc\) perpendicular to \(ab\).

You know by now what we mean by a numerical parallelepiped. It means a parallelepiped whose base is a unit square and whose height is the given number (I mean a line whose ratio to the side of the base of the parallelepiped is the same as the ratio of the given number to one). We extend \(ab\) to \(z\), then construct the parabola \(mbd\), with vertex \(b\), axis \(bz\) and its perpendicular side \(ab\), so the parabola \(mbd\) is known as we have shown previously, and it is tangent to the line \(bc\). We construct a semicircle on \(bc\) which must intersect the (conic) section, say at \(d\).

From \(d\), which is of known location, we draw two perpendicular lines, \(dz\) and \(dh\), to the lines \(bz\) and \(bc\) respectively, so their magnitudes and locations are known. The line \(dz\) is one of the lines of order in the (conic) section, so its square equals the product of (the magnitudes of) \(bz\) and \(ab\). Hence the ratio of \(ab\) to \(dz\) (which is equal to \(bh\)) is the same as the ratio of \(bh\) to \(hd\) (which equals \(zb\)). But the ratio of \(bh\) to \(hd\) is the same as the ratio of \(hd\) to \(hc\). So the four lines \(ab\), \(bh\), \(hd\) and \(hc\) are proportional. Hence the ratio of the square of \(ab\) (the first) to the square of \(bh\) (the second) is the same as the ratio of \(bh\) (the second) to \(hc\) (the fourth). Hence, the parallelepiped, whose base is the square of \(ab\) (which is the number of the roots) and height is \(bh\), equals the (volume of the) cube \(bh\), because their heights are equivalent to their bases. We make the parallelepiped whose base is the square of \(ab\) and height \(bh\), common, so the cube \(bh\) plus this parallelepiped equals the (volume of the) cube whose base is \(ab\) and height is \(bc\), which we assumed to equal the given number.

But the parallelepiped whose base is the square of \(ab\) (which is the number of the roots) and whose height is \(bh\), which is the side of the cube, equals the given number of the sides of the cube \(bh\). Hence the cube \(bh\) plus the assumed number of its sides equals the given number, which is the desired result.
This type has no different cases and none of its problems are impossible to solve. Properties of the circle and the parabola were used.

Second type: Cube plus a number equals sides \((x^3 + a = bx)\)

Assume \(ab\) is a side of a square that is equal (in magnitude) to the number of the roots. We construct a parallelepiped with a square base of side \(ab\), and whose (volume) equals the given number. Let \(bc\) be its height which is perpendicular to \(ab\). We construct the parabola \(dbh\), with vertex \(b\), axis along \(ab\) and its right side is \(ab\). So the location of this parabola is known.

Now, we construct another (conic) section, the hyperbola \(hez\), with vertex at \(c\), its axis along \(bc\), and each of its two sides, the perpendicular and the oblique, equals \(bc\). The location of this hyperbola is known, as was shown by Apollonius in proposition \(nh\) of article \(a\). These two (conic) sections either meet or do not meet. If they do not meet, then there is no possible solution. If they meet tangentially at one point, or they intersect at two points, then the location of these point(s) is (are) known.

Assume they intersect at the point \(h\). We draw two perpendiculars \(ht\) and \(hm\) to the two lines \(bt\) and \(bm\). Surely, the location and magnitude of these perpendiculars are known. Since the line \(ht\) is one of the lines of order, it follows that the ratio of the square of \(ht\) to the product of \(bt\) and \(tc\) is the same as the ratio of the perpendicular side to the oblique one, as was shown by Apollonius in proposition \(k\), article \(a\). The two sides, the perpendicular and the oblique, are equal. Hence, the square of \(ht\) equals the product of \(bt\) and \(tc\), and so the ratio of \(bt\) to \(th\) is the same as the ratio of \(th\) to \(tc\). But the square of \(hm\) (which is equal to \(bt\)) equals the product of \(bm\) and \(ba\), as was shown by proposition \(yb\) of article \(a\) in the book of conic sections.

So the ratio of \(ab\) to \(bt\) is the same as the ratio of \(bt\) to \(bm\) and is as the ratio of \(bm\) (which equals \(ht\)) to \(tc\). Thus, the four lines are proportional. Hence, the ratio of the square of \(ab\) (the first) to the square of \(ht\) (the second) is the same as the ratio of \(ht\) (the second) to \(tc\) (the fourth). So the cube of \(bt\) equals (the volume of) the parallelepiped whose base is the square of \(ab\) and height equals \(ct\).

We make the parallelepiped (whose base is the square of \(ab\) and height \(bc\), which we constructed to equal the given number) common, so the cube \(bt\) plus the given number equals the parallelepiped whose base is the square of \(ab\) and height equals \(bt\), which equals the number of the sides of the cube.

We found that this type has many different cases, some of which have no solution, and the properties of the two conic sections (hyperbola and parabola) were used.
Third type: Cube equals sides plus a number \((x' = ax + b)\)

We set \(ab\) to be a side of a square that (its area) equals the number of the sides. We construct a parallelepiped with base equal to the square \(ab\), and (whose volume) equals the given number. Let its height \(bc\) be perpendicular to \(ab\). We extend \(ab\) and \(bc\), then we construct the parabola \(dbh\), with vertex at \(b\), axis along \(ab\) and its perpendicular side is \(ab\), so its location is known. The conic section is tangent to \(bc\), as was shown by Apollonius in proposition \(lc\) in article \(a\). Then we construct another conic section, the hyperbola \(zbh\), with vertex at \(b\), axis along \(bc\), and each of its sides, the perpendicular and the oblique, equals \(bc\). So the location of this section is known and it is tangent to the line \(ab\). The two conics must intersect. Assume they intersect at \(h\), so the location of \(h\) is known. From the point \(h\), we construct the two perpendiculars \(ht\) and \(hm\), which are of known location and magnitude. The line \(hm\) is one of the lines of order. It follows from the above construction that the square of \(hm\) equals the product of \(cm\) and \(bm\). Hence the ratio of \(cm\) to \(hm\) is the same as the ratio of \(hm\) to \(mb\). But the ratio of \(hm\) (which equals \(ht\)) to \(mb\) (which equals \(ht\), which is a line of order of the other section) is equal to the ratio of \(ht\) to \(ab\) (which is the right side of the conic section).

So the four lines are proportional. Hence the ratio of \(ab\) to \(mb\) is the same as the ratio of \(mb\) to \(ht\), and is as the ratio of \(ht\) to \(cm\).

Now, the ratio of the square of \(ab\) (the first) to the square of \(mb\) (the second) is the same as the ratio of \(mb\) (the second) to \(cm\) (the fourth).

Hence the cube of \(mb\) equals (the volume of) the parallelepiped whose base is the square of \(ab\) and height equals \(cm\) because their heights are equivalent to their bases. But this parallelepiped is equal (in volume) to the (volume of the) parallelepiped whose base is \(ab\) and height equals \(bc\), which we constructed so that (its volume) equals the given number plus (the volume of) the parallelepiped, whose base equals the square of \(ab\) and height \(bm\), which is equal to the assumed number of the sides of the cube \(bm\). So the cube of \(bm\) equals the given number plus the given number of sides, which is the required result.
It is clear that there is only one case in this type, and the case of an impossible solution does not occur. The properties of the parabola and hyperbola were used.

**Fourth type: Cube plus squares equal a number** \(x^3 + ax^2 = b\)

We draw the line \(ab\) to equal the given number of squares, and we construct a cube that is equal (in volume) to the given number. Let \(m\) be its side. We extend \(ab\) and we set \(bt\) equal to \(m\). Then we complete the square \(bt\) in the square \(bdce\). On \(d\) we construct the hyperbola \(hdn\) that does not meet \(bc\) and \(bt\), as was shown in the two figures \(d\) and \(h\) of article \(b\), and in figure \(nt\) of article \(a\). The location of the conic section \(hdn\) is known because the point \(d\) is of known location, and the two lines \(bc\) and \(bt\) are of known locations.

We construct the parabola \(ak\), with vertex at \(a\), axis along \(at\), and its right side is \(bc\). So the location of \(ak\) is known.

The two sections must intersect, say at \(h\), so the location of \(h\) is known. From \(h\), we construct the two lines \(hz\) and \(hl\) which are orthogonal to \(at\) and \(bc\) respectively, so the two lines are of known location and magnitude.

I claim that: it is not possible for the section \(akh\) to intersect the section \(hdn\) at a point, where the perpendicular from that point to the line \(at\) meets the line \(at\) or at a point on the extension through \(t\).

Assume, if possible, it meets \(t\), so its square equals the product of \(at\) and \(tb\) (which equals \(bc\)). But this perpendicular equals the perpendicular on \(dt\), so the square of \(td\) equals the product of \(at\) and \(tb\) (which is also equal to the product of \(bt\) with itself), and that is impossible.

Also it does not meet the line at a point on the extension through \(t\), because in such a case, that perpendicular must be shorter than \(td\), which again is impossible. So the perpendicular must pass through a point between \(a\) and \(t\), say \(z\), and let the perpendicular be \(hz\).
The square of $hz$ equals the product of $az$ and $bc$. Consequently, the ratio of $az$ to $hz$ is the same as $hz$ to $bc$. Further, the rectangle $hb$ equals the rectangle $db$ (in area), as shown by proposition $m$ of article $h$ in the book of conic sections. Hence the ratio of $hz$ to $bc$ is the same as the ratio of $bc$ to $bz$. Thus, the four lines $az$, $hz$, $bc$, $bz$ are proportional. So the ratio of the square of $bz$ (the fourth) to the square of $bc$ (the third) is the same as the ratio of $bc$ (the third) to $az$ (the first). We conclude that the cube $bc$ (which we construct so that it is equal in volume to the given number) equals (in volume) the (volume of the) parallelepiped whose base is the square $bz$ and height $az$. But this parallelepiped whose base is the square $bz$ and height $az$ equals the cube $bz$, and equals the parallelepiped whose base is the square $hz$ and height $ab$.

This parallelepiped (whose base is the square $bz$ and height $ab$) equals the given number of squares. Consequently, the cube $bz$ plus the given number of squares equals the given number, and that is what we wanted to show.

This type has no different cases and always has a solution. The properties of both the parabola and the hyperbola were used.

Fifth type: Cube plus a number equals squares ($x^3 + a = bx^2$)
Assume $ac$ equals the given number of squares. We construct a cube that equals (in volume) the given number. Let its side be $m$.

The line $m$ must be either equal to, greater than or less than $ac$. If they are equal, then the problem has no solution. This is because either the (length of the) side (root) of the required cube is equal to, less than or greater than $m$. If it is equal to $m$, then the product of $ac$ and its square (the square of the required cube) equals the cube of $m$, so that the given number equals the number of the squares and there is no need to add the cube. If the (length of the) side of the cube is smaller than $m$, then the product of $ac$ and its square will be smaller than the given number, so the number of squares is smaller than the given number without adding the cube. Finally, if the (length of the) side is larger than $m$, then its cube will be greater than the product of $ac$ and its square without adding the number.
Further, if \( m \) is greater than \( ac \), then it is even more clear that for the above three cases, there is no possible solution. So \( m \) must be less than \( ac \) or no solution will exist for the problem.

Cut a segment \( bc \) from \( ac \) of length equal to the length of \( m \). The line \( bc \) is either equal to, lesser than or greater than \( ab \). Let it be equal to \( bc \), in the first figure, greater than in the second and lesser than in the third.

In the three figures, complete the square \( dc \) and construct a hyperbola which does not meet \( ac \) and \( ch \). The section is \( dz \) in the first and \( dt \) in the second and third. We then construct a parabola with vertex \( a \), axis along \( ac \) and its right side equals \( bc \). It is \( at \) in the first, \( al \) in the second and \( ak \) in the third.

The positions of the two conic sections are known. In the first figure, the parabola will pass through the point \( d \) because the square of \( db \) equals the product of \( ab \) and \( bc \), so \( d \) is on the parabola and they (the two conic sections) meet at some other point, as you may easily discover. In the second, the point \( d \) falls outside the parabola because the square of \( db \) is greater than the product of \( ab \) and \( bc \). So, if the two sections meet tangentially at some other point, or if they intersect, then the perpendicular from this point must meet a point between \( a \) and \( b \) and a solution is possible; otherwise, no solution exists.

The honorable geometer Abu-Aljood was not aware of the case of intersection or tangency, which led him to claim that if \( bc \) was greater than \( ab \), then the problem would be impossible. His claim was wrong.

Al-Mahani was forced to study this one of the six known types.

In the third, the point \( d \) is inside the parabola, so the two conic sections will meet at two points. In summary, we draw from the point of intersection a perpendicular to \( ab \), call it in the second figure \( zt \). Another perpendicular (from the same point \( t \)) is drawn to \( ch \), let it be \( tk \). The rectangle \( tc \) equals the rectangle \( dc \). So the ratio of \( zc \) to \( bc \) is the same as the ratio of \( bc \) to \( tz \). The line \( zt \) is one of the lines of order in the conic section \( atl \). Hence, its square is the same as the product of \( az \) and \( bc \). Consequently, the ratio of \( bc \) to \( tz \) is the same as the ratio of \( zt \) to \( za \). The four lines are proportional: the ratio of \( zc \) to \( ch \) is as the ratio of \( ch \) to \( zt \) and as the ratio of \( zt \) to \( za \). So the ratio of the square of \( zc \) (the first) to the square of \( be \) (the second) is the same as the ratio of \( bc \) (the second) to \( zt \) (the fourth). Hence the cube of \( bc \) (which is equal to the given number) equals the parallelepiped whose base is the square \( zc \) and height \( za \).

We make the cube \( zc \) common. So the cube \( zc \) plus the given number equals the parallelepiped whose base is the square \( zc \) and height \( ac \), which equals the number of squares. This is the required result.

The other two cases are similar, except in the third case where we need to draw two cubes, because each perpendicular cuts one side of the cube from \( ca \), as shown before.

It was shown that for this type of equation there are different cases, some of which have no solution. Properties of the two conic sections, hyperbola and parabola, were used.
Sixth type: Cube equals squares and numbers \((x^3 = ax^2 + b)\)

Assume the number of the squares equals the length of the line \(ab\). We construct a parallelepiped with height \(ab\) and square base (whose volume) equals the given number. Let its side \(bc\) be perpendicular to \(ab\). We then complete the surface (rectangle) \(db\). We draw at the point \(c\) the hyperbola \(c_hz\), which does not meet \(ab\) or \(ad\). We draw another (conic) section, the parabola \(bhm\), with vertex \(h\), axis along \(ab\), and right side \(ab\). These two sections must intersect, say at \(h\). So the location of \(h\) is known. We draw from \(h\) two perpendiculars, \(ht\) and \(hk\), on \(ab\) and \(ad\) respectively. So the (area of) rectangle \(ha\) equals the rectangle \(ca\). Hence the ratio of \(ak\) to \(bc\) is the same as the ratio of \(ab\) to \(hk\), and their squares are also proportional. But the square of \(hk\) equals the product of \(kb\) and \(ab\) because \(hk\) is one of the lines of order of the (conic) section \(bhm\). So the ratio of the square \(ab\) to the square \(hk\) is the same as the ratio of \(ab\) to \(bk\). Hence, the ratio of the square of \(bc\) to the square of \(ak\) is the same as the ratio of \(bk\) to \(ab\). Thus the parallelepiped whose base is the square \(bc\) and height \(ab\) equals (in volume) the parallelepiped whose base is the square \(ak\) and height \(kb\) because both of their bases and heights are equivalent. We make the parallelepiped with base the square \(ab\) and height \(ab\) common. So the cube \(ak\) equals the parallelepiped (whose base is the square \(bc\) and height \(ab\) which we constructed to equal the given number) plus the (volume) parallelepiped whose base is the square \(ak\) and height \(ab\) which equals the assumed number of squares. Consequently, the cube \(ak\) equals the given number of squares plus the given number.

This type has no different cases and a solution is always possible. The properties of the two (conic) sections, parabola and hyperbola, were used.
Third degree equations of four terms each, part 1

Since we have finished the study of equations of three terms each, we move now to the four types of four terms each; all have three terms equal to one term.

First type: Cube plus squares plus sides equals a number \((x^3 + ax^2 + bx = c)\)

We draw \(bh\) a side of a square which is equal to the assumed number of sides. Then, we construct a parallelepiped whose base is the square \(bh\) and (whose volume) equals the given number. Let its height \(bc\) be perpendicular to \(bh\). We construct \(bd\) along \(bc\), to equal the given number of squares. On the diameter \(dc\) we construct the semicircle \(dzc\), and we complete the rectangle \(bk\). Then, we construct a hyperbola with vertex at \(c\) which does not meet \(bh\) or \(hk\). So the parabola must intersect the circle at a point \(z\), because it intersects its tangent line \(ck\). Hence the section must intersect the circle at a second point, say at \(z\). So the location of \(z\) is known because the locations of both the circle and the (conic) section are known.

From the point \(z\) we draw two perpendiculars, \(zt\) and \(za\), to \(hk\) and \(ha\) respectively. The rectangle \(zh\) equals the rectangle \(bk\). Removing the common rectangle \(hl\), we are left with two equal rectangles, \(zb\) and \(lk\). So the ratio of \(zl\) to \(lc\) is the same as the ratio of \(hb\) to \(bl\) because \(hb\) equals \(tl\); further, their squares are also proportional. But the ratio of the square of \(zl\) to the square of \(lc\) is the same as the ratio of \(dl\) to \(lc\) – for the circle – so the ratio of the square of \(hb\) to the square of \(bl\) is the same as the ratio of \(dl\) to \(lc\). Hence, the parallelepiped (whose base is the square \(hb\) and height \(lc\)) equals (in volume) the parallelepiped whose base is the square \(bl\) and height \(dl\). But the second parallelepiped equals (in volume) the cube \(bl\) together with the parallelepiped whose base is the square of \(bl\) and height \(bd\), which equals the given number of squares.

We make the parallelepiped (whose base is the square of \(hb\) and height equals \(bl\), which equals the number of the roots) common. So the parallelepiped whose base is the square \(bh\) and height \(bc\), which we constructed to equal the given number, will equal the cube of \(bl\) plus the number of the given sides plus the given number of squares. That is what we wanted to show.

This type has no different cases and always has a solution. Properties of circles and hyperbolas were used.
Second type: Cube plus squares plus numbers equals sides ($x^3 + ax^2 + b = cx$)

Construct $ab$ to be the side of a square which is equal to the number of sides and $bc$ to equal the given number of squares; further, $be$ is perpendicular to $ab$. We construct a parallelepiped whose base is the square with $ab$ and is equal (in volume) to the given number. Let its height $bd$ be along the extension of $be$. At the point $d$, we construct the hyperbola $zdm$ that does not meet $ab$ or $ah$ after we complete the rectangle $bh$. We construct another hyperbola, $tdm$, with vertex $d$, axis along the extension of $bd$ and each of its two sides, the normal and the oblique, equals $dc$. No doubt, this section must intersect the first at $d$. If they meet at a second point, then the problem has a solution; otherwise it is impossible to solve the problem. Such a meeting of the two sections, whether by tangency at one point or by intersection at two points, is based on article four of the book of conics. We promised to refer to only two articles from that book. But once they meet, it does not matter how they meet, whether by tangency or by intersection. If they intersect at a point other than $d$, they must intersect at two points. In either case (tangency or intersection), we draw from the point $m$ (the point of intersection or the point of meeting, as the case may be) two perpendiculars, $mn$ and $kml$. Since the point $m$ is of known location, both perpendiculars are of known location and magnitude.

The rectangle $am$ equals (in area) the rectangle $ad$. We remove the common rectangle $hn$, and we get $nd$ equals $hm$. We make $dm$ common. It follows that $nl$ equals $hl$ because their sides are equal and so are the squares of their sides. Hence, the ratio of the square of $ab$ to the square of $bl$ is the same as the ratio of the square of $ml$ to the square of $ld$.

But the ratio of the square of $ml$ to the square of $ld$ is the same as the ratio of $cl$ to $ld$ as we have shown repeatedly. Consequently, the ratio of the square of $ab$ to the square of $bl$ is the same as the ratio of $cl$ to $ld$. Thus, the parallelepiped – whose height is $ld$ and base equals the square of $ab$ – equals the parallelepiped whose base is $bl$ and height $lc$. But the latter parallelepiped equals the cube of $bl$ plus the parallelepiped whose base is the square of $bl$ and height $bc$, which equals the given number of squares.

We make the parallelepiped (whose base is the square of $ab$ and height $bd$, which was constructed to equal the given number) common. Hence, the cube $bl$ plus the number of squares plus the given number, equals the parallelepiped whose base is the square of $ab$ and height $bl$ (which equals the given number of sides of the cube of $bl$). This is the required result.

It is clear that this type has many different cases, including the case of two different roots for two different cubes. There might be cases that are impossible to solve – I mean the problem associated with the case is impossible to solve. Properties of two hyperbolas were used.
Third type: Cube plus sides plus numbers equals squares ($x^3 + ax + b = cx^2$)

We draw the line $bh$ to equal the given number of squares and $bc$, a side of a square that is equal to the assumed number of sides, perpendicular to $bh$. We construct a parallelepiped whose base is the square of $bc$ and is equal (in volume) to the given number. Let its height be $ab$ on the extension of $bh$.

On $ah$ we construct a semicircle $azh$. The point $c$ is either inside the circle, on its circumference, or outside it. Let it fall first inside the circle.

We extend $bc$ until it intersects the circle at $z$, then we complete the rectangle $ac$. Then, on $zc$, we construct the rectangle $zm$ that is equal (in area) to the rectangle $ac$. The location of $m$ is known because the position, magnitude and angles of the surface $cm$ are known, and the location and the magnitude of the line $zc$ are known. The point $m$ must be inside the circle, on the circumference, or outside it.

Assume first that it falls inside the circle. We construct a hyperbola with vertex $m$, which does not meet $zc$ and $zn$. Hence it must intersect the circle at two points, say $l, n$, whose locations are known. From these two points we draw the two lines $lk$ and $nf$ perpendicular to $ah$. Further, from $l$ we draw the perpendicular $lt$ to $bz$.

The rectangle $lc$ equals (in area) the rectangle $cm$, and the rectangle $cm$ equals $ca$. We make $ck$ common, so $dk$ equals $tk$, because their sides are equivalent and so are the squares of their sides. But the ratio of the square of $lk$ to the square of $ka$ is the same as the ratio of $hk$ to $kl$. It follows that the ratio of the square of $bc$ to the square of $bk$ equals the ratio of $hk$ to $ka$. Hence, the parallelepiped whose base is the square $bc$ and height $ka$ equals (in volume) the parallelepiped whose base is the square of $bk$ and height $kh$. But the first parallelepiped equals the given number of sides of the cube $bk$ and equals the given number. We make the cube $bk$ common. Consequently, the parallelepiped (whose base is the square $bk$ and height $bh$, which equals the given number of squares of the sides of the cube $bk$) equals the cube $bk$ plus the given number of sides plus the given number.

A similar result holds for the cube $bf$, if the points $c$ and $m$ fall inside the circle.
If \( m \) falls outside the circle, then we construct the hyperbola with vertex at \( m \). The section either intersects the circle or it meets the circle tangentially. In either case, it will be similar to what we have already mentioned.

The variety of this type was mentioned by Abu-Aljood in solving the problem which we will discuss.

If the section does not meet the circle, we construct a surface on a line shorter than \( zc \) or longer, when \( m \) lies inside the circle. If the section in such a case does not meet the circle then no solution is possible. The proof that no solution is possible is the opposite of what we have mentioned.

If \( c \) falls on the circumference of the circle or outside it, then we extend \( cz \) and construct a rectangle, with one of its angles at \( c \), in such a way that if we construct at the opposite angle to \( c \) a section as described before, it will meet the circle by intersection or tangency. This can be proved by a simple deduction that I have left as a mathematical exercise for the reader of my essay. Whoever could not prove the deduction will gain nothing from this essay, it being based on the previously mentioned three books. We prove that the impossible is not possible by reversing the proof that we presented for the possible. This is based on the fact that the side of the cube must be shorter than \( bh \), which equals the given number of squares. For if the side of the cube is equal to the assumed number of squares, then the cube is equal (in volume) to the assumed number of squares plus some part of the number and the sides.

If the side of the cube is larger than the given number of squares, then the cube itself will be larger than the given number of squares plus something else added to it. So the side of the cube has to be smaller than \( bh \).

We separate a part from \( bh \) that is equal to the side of the cube, let it be \( bf \). At \( f \), we construct a perpendicular that meets the circumference of the circle, then we reverse the above-mentioned proof. This shows that the tip of the perpendicular is on the circumference of the conic section, whereas we said it does not meet the circle, and this is impossible.

Since we believe that this deduction might be difficult for some readers of this essay, we leave this argument and present a formula instead of this deduction.

We construct, in any way we want, a rectangle on the line of extension of \( bc \) regardless of the location of \( c \), inside or outside the circle. We make such a rectangle so that one of its angles is at \( c \) and it is equal to the rectangle \( ac \). Certainly, the sides of this rectangle are of known location and magnitude.

On the opposite angle of \( c \), we construct a hyperbola that does not meet \( zc \) and \( cn \) (which is perpendicular to \( zc \) at \( c \)). If the section meets the circle by intersection or tangentially, then a solution is possible, otherwise it is impossible. The proof of the impossibility is as we have already mentioned.

One of the geometers needed this type in his work, and succeeded in proving it, but he did not show the different cases that may occur and he did not realize that there is a case where the solution is impossible as we have mentioned. So make sure you understand this type and the last formula in the proof of this type together with the distinction between the possible and the impossible cases.
This type uses the properties of the circle and the hyperbola, and this is what we needed to show. But the problem that led one of the late geometers to this type is: divide ten into two parts so that the sum of the squares of the two parts plus the outcome of dividing the larger part by the smaller one equals seventy-two.

He denoted one part by \( x \), and the other by \( 10 - x \), as algebraists usually do in such divisions. This division leads to: cube plus five plus thirteen and a half of the side of the cube equals ten squares. In this specific problem, the two points \( c \) and \( m \) will fall inside the circle.

This honorable man was able to solve this problem, though a number of honorable men from Iraq could not solve it, among them was Abu Sahel Al-Qohi. But this man (who solved the problem) despite his great knowledge of mathematics, did not realize all the different cases, and that some cases are impossible to solve. This honorable man is Abu-Aljood, nick-named Al-Shanni, and God knows better.

Fourth type: Numbers plus sides plus squares equals a cube

\( a + bx + cx^2 = x^3 \)

Assume \( bh \) to be a side of a square that is equal to the assumed numbers of sides. Construct a parallelepiped whose base is the square \( bh \) and (whose volume) equals the given number. Let its height \( ab \) be perpendicular to \( bh \). We let \( bc \) be along \( ab \) and is equal to the assumed number of squares. Then we complete \( ah \) and we extend \( hb \) along \( hm \) regardless of the magnitude. On the given \( hm \) we construct the surface \( hn \) that equals \( ah \). So the location of the point \( n \) is known. At \( n \), we construct the hyperbola \( ntk \) that does not meet \( hm \) and \( hs \), and so it is of known location.

We construct another hyperbola, \( lct \), with vertex at \( c \), and axis along \( bc \), and each of its sides, the perpendicular and the oblique, equals \( ac \). So the location of
this hyperbola is known. Further, it must meet the section \(ntk\), say at \(t\), so \(t\) is of known location.

From \(t\), we construct the two perpendiculars \(td\) and \(tp\) to \(bc\) and \(bm\). The locations and magnitudes of such perpendiculars are known. Further, \(th\) equals \(hn\) which in turn equals \(ha\). We make \(hp\) common, so \(as\) equals \(tb\) because their sides are equal (in length) and so are the squares of their sides. But the ratio of the square of \(tp\) to the square of \(ap\) is the same as the ratio of \(pc\) to \(ap\), as we have shown many times for the section \(lct\).

Now, the ratio of the square of \(bh\) to the square of \(bp\) is the same as the ratio of \(pc\) to \(pa\). Thus, the parallelepiped whose base is the square \(bh\) and height \(ap\) equals the parallelepiped whose base is the square \(bp\) and height \(cp\). But the first parallelepiped equals the parallelepiped which we constructed to be equal to the assumed number plus the parallelepiped whose base is the square \(bh\) and height \(bp\) (which is equal to the given number of the sides of the cube \(bp\)).

![Diagram of geometric figures](image)

We make the parallelepiped, whose base is the square \(bp\) and height \(bc\) (which is equal to the given number of squares of the cube \(bp\)) common. So the cube of \(bp\) is equal to the given number of squares plus the given number of sides plus the given number. This is the needed result. This type has no different cases and always has a solution.

**Third degree equations of four terms each, part 2**

Since we finished the four types each of which consists of four terms, we move on to the three types each of which consist of two terms equals two terms.
First type: Cube and squares equal sides and numbers

We make $bd$ a side of a square which equals (in area) the assumed number of sides, and construct $cb$ to equal the assumed number of squares and be perpendicular to $bd$. We construct a parallelepiped with base $bd$ and volume equal to the assumed number; let its height be $s$. So the line $s$ is either greater than, smaller than or equal to $bc$.

First, let it be smaller. We cut from $bc$ the segment $ab$ that equals $s$, and we complete the square $ad$. Assume $dz$ is an extension of $bd$ regardless of magnitude. We construct a rectangle on $dz$ that is equal to $ad$, call it $hd$. So the location of $h$ is known and the sides of the rectangle $hd$ are all of known location and magnitude. At $h$, we construct the hyperbola $hm$, that does not meet $zd$ or $dy$. So the location of $hm$ is known. We construct another hyperbola $amt$, with vertex at the point $a$ and axis $ab$, and each of its sides, the perpendicular and the oblique, equals $ac$. This hyperbola must intersect the other section. Let the two sections intersect at $m$, so the location of $m$ is known. From $m$ we draw two perpendicularears $mk$ and $ml$ whose location and magnitude are known. The rectangle $md$ equals the rectangle $hd$ which in turn equals the rectangle $ad$, and $dk$ is common. The surface (rectangle) $hb$ equals (in area) the rectangle $am$; so their sides are equivalent and hence the squares of their sides are equivalent. But the ratio of the square of $mk$ to the square of $ka$ is the same as the ratio of $ck$ to $ak$, as we have shown many times.

Hence, the ratio of the square of $bd$ to the square of $kb$ is the same as the ratio of $ck$ to $ak$. The parallelepiped, whose base is the square of $bd$ and height $ak$, is the same as the parallelepiped whose base is the square of $bk$ and height $ck$. But the latter parallelepiped equals (in volume) the cube $bk$ plus the parallelepiped whose base is $bk$ and height $ck$ which equals the given number of squares.

The first parallelepiped equals the parallelepiped whose base is the square of $bd$ and height $ab$, which we constructed to equal the given number plus the (volume) parallelepiped whose base is the square $bd$ and height $bk$, which equals the given number of sides of the cube $bk$.

So the cube of $bk$ plus the given number of its squares equals the given number plus the given number of sides. This is the required result.
If $s$ equals $bc$, then $bd$ is the side of the required cube.

**Proof:** The solid (parallelepiped) whose base is the square of $bd$, and whose height also equals $bd$ (which equals the number of sides of the cube $bd$) equals (in volume) the cube of $bd$.

The cube, whose base is the square of $bd$ and whose height is $bc$ (which equals the given number of squares of the cube of $bd$), equals the parallelepiped whose base is the square of $bd$ and height $s$ (which equals the assumed number). So the cube of $bd$ plus the given number of squares equals the assumed number plus the assumed number of sides, and this is the desired result.

In such a case, it is known that the cube of $bd$ plus the assumed number equals the assumed number of squares (of the cube of $bd$) plus the given number of sides. There is an intersection between this type and the third type, which is: cube and numbers equal squares and sides.

If $s$ is greater than $bc$, then we make $ab$ equal to $s$, and construct the second conic section at the point $c$ such that each of its two sides equals $ac$. This second section must intersect the other section, and the side of the cube will again be $bk$. The rest of the construction and proof is similar to the previous one except that the ratio of the square of $mk$ to the square of $ka$ is the same as the ratio of $ak$ to $kc$.

We found that this type has different cases and types; one of its two cases has much in common with the third type and each of the cases has a solution, and it uses the properties of two conic sections.

**Second type: Cube plus sides equals squares and numbers**

We set $bc$ to equal the number of the given squares. We draw $bd$ to be a side of a square that is equal (in area) to the number of sides, and is perpendicular to $bc$. Then, we construct a parallelepiped which is equal (in volume) to the given number, such that its base is the square of $bd$. Let its height be $s$. The line $s$ is either smaller than, equal to or larger than $bc$. 
Assume first that \( s \) is smaller than \( bc \). We cut from \( bc \) the segment \( ha \) which equals \( s \). Then we form the rectangle \( ad \), and construct the circle \( akc \), with diameter \( ac \). The location of the circle is known. At the point \( a \), we construct a hyperbola which does not meet \( bd \) or \( dz \), call it \( mat \). Again its location is known and \( mat \) will intersect \( az \), the tangent to the circle. So it must intersect the circle, because if it falls between the circle and \( az \) we could draw from the point \( a \) a line which is tangent to the section as was shown by Apollonius in diagram \( s \) in his article \( h \). That line either falls between \( az \) and the circle, which is impossible, or it falls outside \( az \), so \( az \) would be a straight line between the conic section and the tangent line, and that is impossible too. So the conic section \( tam \) does not fall between the circle and \( az \). So it must intersect the circle, and it must intersect it at another point, call it \( k \), whose location is known. We draw two perpendiculars from \( k \), say \( kn \) and \( kh \) to \( bc \) and \( bd \). Both the location and magnitude of \( kn \) and \( kh \) are known. We complete the rectangle \( kd \).

The rectangle \( ad \) equals the rectangle \( kd \). We remove the common square \( nz \) and we make \( ak \) common. So \( bk \) equals \( al \) since their sides are equivalent, and hence the squares of their sides are equivalent. But the ratio of the square of \( kh \) to the square of \( ha \) is the same as the ratio of \( he \) to \( ha \). It follows that the ratio of the square of \( bd \) to the square of \( bh \) is the same as the ratio of \( ch \) to \( ha \). Hence, the parallelepiped whose base is the square of \( bd \) and height \( ha \), equals (in volume) the parallelepiped whose base is the square of \( bh \) and height \( ch \).

We make the cube \( bh \) common, so the parallelepiped whose base is the square of \( bh \) and height \( bc \), equals the cube of \( bh \) plus the volume of the parallelepiped whose base is the square of \( bd \) and height \( ha \). But the first parallelepiped equals the given number of squares of the cube \( bh \).

We set the parallelepiped – whose base is the square \( bd \) and height \( ha \) which we constructed to equal the given number – common. So the cube \( bh \) plus the (volume) parallelepiped – whose base is the square \( bd \) and height \( bh \) which equals the given number of sides of the cube \( bh \) – equals the given number of squares plus the given number, and this is the desired result.
If $s$ equals $bc$, then $bc$ is the side of the required cube.

**Proof:** The cube of $bc$ equals the given number of its squares. And the parallelepiped, whose height is $bc$ and whose base is the square of $bd$, equals the given number, and it also equals the assumed number of the sides of the cube $bc$. So the cube $bc$ plus the given number of its sides equals the given number of squares plus the given number.

This type has intersection with the third kind, because the given number of the sides of the cube $bc$ equals the assumed number. It follows that the cube of $bc$ plus the given number equals the given number of its squares plus the given number of sides.

If $s$ is larger than $bc$, we set $ba$ equal to $s$ and construct the circle with diameter $ac$, and the conic section at $a$ intersects the circle at $k$ as we have shown. From $k$ we draw two perpendiculars $kh$ and $kn$ as we did in the previous figure, so $bh$ is the side of the required cube. The proof of this claim is as before.

We omit the common rectangle $hd$, so the sides $hn$ and $hz$ are equivalent and their squares are equivalent. The rest of the proof is exactly as the previous one.

We saw that this kind has different cases and types. One of them has some intersection with the third type and none of its cases are impossible to solve. The properties of circles and hyperbolas were used.

Third type: Cube and numbers equals sides and squares
Assume that $bc$ equals the assumed number of squares. Let $bd$ be perpendicular to $bc$ and a side of a square that equals the assumed number of sides. Construct a parallelepiped whose base is the square of $bd$ and is equal to the given number. Let $s$ be its height. So the line $s$ is smaller than, equal to, or larger than $bc$.

First, let it be smaller. We cut from $bc$ the segment $ba$ that equals $s$, and then we complete the rectangle $bz$. At $a$, we construct the hyperbola $mat$, which does
not meet \(bd\) or \(dz\). We construct another hyperbola, call it \(kcl\), with vertex at \(c\), axis along the extension of \(bc\), and each of its two sides, the perpendicular and the oblique, equals \(ac\). So it must intersect the other section. Assume that the section \(kcl\) and the section \(mat\) intersect at \(p\). So the location of \(p\) is known because the two sections are of known locations. From \(p\) we construct two perpendiculars \(pn\) and \(hpu\), whose locations and magnitudes are known. The rectangle \(da\) equals the rectangle \(dp\), so \(nh\) equals \(zh\) as we have shown many times. Their sides are equivalent and so are the squares of their sides. But the ratio of the square of \(ph\) to the square \(ha\) is the same as the ratio \(ch\) to \(ha\). So the ratio of the square of \(bd\) to the square of \(bh\) is the same as the ratio \(ch\) to \(ha\). The parallelepiped whose base is the square of \(bd\) and height \(ha\) equals (in volume) the parallelepiped whose base is the square of \(bh\) and height \(ch\).

![Diagram](image)

We make the parallelepiped – whose base is the square of \(bh\) and height \(bc\), which equals the number of the squares of the cube \(bh\) – common. The cube of \(bh\) equals the assumed number of the given squares plus the (volume) parallelepiped whose base is the square of \(bd\) and height \(ha\). We then make the parallelepiped – whose height is \(ha\) and base the square of \(bd\) which we constructed to equal the given number – common. Hence, the parallelepiped – whose base is the square of \(bd\) and height \(bh\), which equals the number of the sides of the cube of \(bh\) plus the number of the squares of the cube \(bh\) – equals the cube of \(bh\) plus the given number.
If $s$ equals $bc$ then $bc$ is the side of the (required) cube.

**Proof:** The cube of $bc$ equals the assumed number of its squares. The assumed number equals the assumed number of the sides of the cube of $bc$. So the cube of $bc$ plus the given number equals the given number of its squares plus the given number of its sides, which is the claim. Also, the cube of $bc$ plus the given number of its sides equals the given number of its squares plus the given number. This type has some intersection with the second kind.

If $s$ is larger than $bc$, we make $ba$ equal to $s$ and then complete the rectangle. On $a$, we construct the two sections (the first and the second). They must intersect. They meet at one point if they are tangent to each other, or at two points if they intersect, as known from article $d$ of the book of conic sections. The problem either has a solution or not. If the conic sections intersect, we draw from the two points of intersection two perpendiculars that separate two sides of two cubes. The proof is as the previous one with no change. This type has many kinds; some have no solution and the properties of two conic sections were used.

And it was clear that the three quartile types have common cases. That is to say, there is a kind of the first that is exactly a kind of the second, and a kind of the second that is a kind of the third, and a kind of the third that is exactly the same as a kind of the second, as shown.

*Equations that contain the inverse of the unknown*

We have studied the twenty-five types of the introductory algebra and equations and covered them completely. We discussed the different kinds of each type. We also presented rules of how to distinguish the ones with possible solutions from those which are impossible to solve. We showed that most of them have solutions. Let us now study their parts (inverses).
The part of a number (its reciprocal) is a number whose ratio to one is the same as the ratio of one to the number. So if the number is three, then its part is one third, and if the number is one third, than its part is three. Similarly, if the number is four, then its part is one fourth, and if it is one fourth then its part is four. In summary, the part of a number is the part called by the same name as the number, such as one third to three if the number is an integer, and three to one third if it is a rational number. Similarly, the part of a square is the part called by the same name whether it be an integer or a rational. Likewise for the part of a cube. To make things more clear, let us put it in a table.

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<th>Part of cube</th>
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<th>Cube</th>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

So the ratio of the part of the cube to the part of the square equals the ratio of the part of the square to the part of the root; and also equals the ratio of the part of the root to one, and equals the ratio of one to the root; and equals the ratio of the root to the square and equals the ratio of the square to the cube. These are seven ranks in a sequence, with the same ratio. We will talk about their equations only.

The part of square of the square and the part of square of the cube and the part of cube of the cube, as far as we can go, will also be in ratio and we do not need to mention them, for there is no way to deduce it.

Let it be known that if you take one eighth – which is part of cube – as a cube then its part is eight, which is a cube and vice versa.

Take this as a model for the rest. The part of the cube and the part of the square and the part of the root and one; these four have the same status as the cube, square, root and one.

**Example:** If we say: part of a square equals half part of a root, then this is equivalent to saying: a square equals twice a root. So the square is one quarter which is a part of a square. Hence, the required square is four and its part is one fourth and the part of its root is half and so on.

As for the complex ones (multi-term ones): If one says: part of a square plus two parts of a root equals one plus one fourth, then this is equivalent to: a square plus two roots equals one plus one fourth. Using the same method we used for the simple cases above, we get the root equals half and the square equals one fourth. But according to the question: part of a square plus two parts of a root
equals one plus one fourth, we get the fourth (which is the first square) is the part of the required square, and so the required square is four.

Also, in quartiles (equations with four terms): If one says: part of a cube plus three parts of a square plus five parts of a root equals three plus three eighths, this is the same as saying: a cube plus three squares plus five roots equals three and three eighths.

Using the method that we explained via conic sections, we can determine the side of a cube, which will be the part of the required root. We set its ratio to the assumed one to equal the ratio of the assumed one to another line, so such a line is the side of the required cube. So it seems that there are twenty-five more types of these equations among these four, comparable with the previous twenty-five types.

As for multiplying them with each other, this is discussed in the books of algebraists, and one can deduce it easily, so there is no need to go into details. As for relating these four (equations) with the previous four, I say: If we say: a cube equals ten parts of a cube (that is: ten parts of itself) then the cube would be the first in seven ranks, and the parts of the cube are the seventh of them. So multiply one by the other and take the root of the outcome; the result is the middle term, I mean the fourth, which is the required cube.

To see this: Notice that if any number is multiplied by its reciprocal, the result is one. If it is multiplied by twice its reciprocal, the result is two. And if it is multiplied by ten times its reciprocal, the result is ten times the number. This is equivalent to the question: “what cube when multiplied by itself gives the product ten?” So its root is the required cube, and finding the side of that cube is done as we have shown using conic sections.

Similarly, the question: “what square equals sixteen of its reciprocal?” To solve it, multiply the reciprocal by sixteen and take the root of the outcome, which equals four, that would be the required square. This is equivalent to the question: “what square when multiplied by itself equals sixteen?”

Similarly, the question: “what root equals four of its reciprocal?” is equivalent to the question: “what number multiplied it by itself gives four?”, and the answer as we know is two.

However, a deduction of a solution to the question: “what square equals a number of parts of the side of its cube?” is not possible using the previous methods. This is because we need to introduce four lines between two lines, in such a way that the respective ratios (of their lengths) of the six lines are the same. This was deduced by Abu Ali Ibn Al-Haitham, but it is very difficult and cannot be included in this book.

Similarly for the question: “what cube equals a number of parts of the square of its side?” It needs the above construction which could not be solved by our methods.

In summary, multiplying the first by the sixth of these seven ranks needs the introduction of four lines between two lines so that the respective ratios (of their lengths) of the six lines are the same, as was shown by Abu Ali Ibn Al-Haitham.
But to solve the question: “what cube equals sixteen parts of its side?” we multiply the first by the fifth, so the root of the root of the number equals the side of the required cube. The same method is used for any of the seven ranks that equals its fifth in the ratio within the seven ranks.

In the complex ones (multi-term equations), the statement: “a root equals one and two parts of the root”, is equivalent to: “a square equals a root and two”. This is because these three are in ratio with the previous mentioned three, and we solve it using the above-mentioned method. We get the square equals four which equals its root and two. The root of this is the needed number. Its root is two, which equals one and two parts of its root.

Similarly, “a square plus its two roots equal one plus two parts of a root”, is equivalent to “a cube and two squares equal a root and two”. We find the side of the cube using the method of conic sections, so the square of that side is the needed square.

Also, “a root plus two plus ten parts of the root equals twenty parts of a square” is equivalent to “a cube and two squares and ten roots equal twenty”. So we get the side of the cube using conic sections and it will be the required root.

In summary, any four successive ranks of these seven ranks would fall in the same category of the mentioned twenty-five types. If it exceeds five, six or seven ranks, then there is no way to solve it. For example, the equation: a square plus two roots equals two plus two parts of the square, cannot be solved, because the square is the second rank and the part of the square is the sixth, and that exceeds five ranks. We use the same ideas for the other cases.

The odd types among these seven ranks amount to twenty-one types, two of which cannot be solved using our methods, and they need the argument of Ibn Al-Haitham. So, we are left with nineteen types that can be solved using our methods. We need to use the properties of the circle to solve some, and the properties of conic sections for others.

All the sequential triplicates constitute fifteen types, and they can be solved using the properties of the circle. The triplicate ones in every four consecutive ranks constitute twenty-four types, and can be solved using conic sections. Further, all the quartiles among every four consecutive ranks constitute twenty-eight types, and can be solved using properties of conic sections.

So all the types (which fall within these seven ranks that can be solved using our methods) amount to eighty-six; only six were mentioned in the books of ancient algebraists.

Anyone who read these introductions and had incentive and insight into these problems could have discovered what the ancients could not. It is time to end this essay with gratitude to God, and praising all of His prophets.

**Abu-Aljood Ben Al-Laith problem**

Five years after I finished this article, I was told by some people who knew little of geometry, that the geometer, Abu-Aljood Mohammad Ben Al-Laith,
has done some work on the classification of these types; that he solved most of them using conic sections, but did not cover all types and did not classify them as solvable ones and the ones with no possible solution; he only discussed minor problems.

I thought that might be possible because these two types (of equations) which I attributed to someone else, are attributed to him. And I read them in a collection of classifications for Abu-Aljood with the handwriting of Al-Hazimi Al-Khawarazmi, including a three-term equation: cube plus a number equals squares. It has different kinds, with certain conditions on each kind, as was mentioned in this article. Some of these conditions were not fulfilled in his setup. He claimed that there is no solution of this kind, saying: if the side of the cube that equals the assumed number is larger than half the number of the squares, then the problem has no solution, which is not true as we have shown. The reason (behind his false claim) is that he was not aware of the case that the two conic sections may meet tangentially or by intersection.

A second type (of equations) that Abu-Aljood discussed is a quartile (four-term equation): a cube plus a number plus sides equals squares. I acknowledge that he studied this problem thoroughly after many geometers failed to solve it, though the problem was a minor one.

This type has different kinds and there are conditions to be fulfilled. Some of its (this type) problems have no solution, and Abu-Aljood did not study them thoroughly. I mention all of this so that any person who has a chance to read both articles could compare my article with this honorable person's article and see if what I was told about this honorable person is correct.

I believe I did my best to study all problems thoroughly but with short proofs and without unnecessary details.

I could have presented an example for each of the types and kinds (of problems) to prove validity. But I avoided lengthy arguments and I limited myself to general rules counting on the intelligence of the student, because any person who has enough intelligence to understand this article would not fail to produce what he needs of partial examples. God gives guidance to the good, and on Him we depend.

One of my friends suggested that I should show the flaw in the proof of Abu-Aljood Mohammad Ben Al-Laith of the fifth type of the six triplicate (three terms) types that can be solved by conic sections which is: a cube plus a number equals squares.

Abu-Aljood said: We set the line $ab$ to equal the number of squares, and we cut from $ab$ the segment $bc$ to be a side of a cube that equals the assumed number. So the line $bc$ is either equal to, greater than, or smaller than $ca$.

He said: If they are equal, we complete the rectangle $ch$ then construct a hyperbola at $d$ that does not meet $ab$ and $bh$. Further we construct a parabola with vertex at $a$ and axis $ab$ and right side $bc$. This conic section will pass through the point $d$ as we have shown. He claimed that the two sections will be tangent to each other at $d$. This is false, because they must intersect.

**Proof:** We make $bz$ equal to $ba$, then draw $az$. It must pass through $d$. 


Further, it will be inside the parabola, the angle adb will be a right angle and the two angles abd and zbd are equal. It is known that the axis of a parabola divides the angle of the conic into halves. So the line bdt must be the axis of the hyperbola at d, and the line ad is parallel to the lines of order, so it is tangent to the hyperbola.

Consequently, the parabola must intersect the hyperbola and not be a tangent to it, for if it were a tangent, then the lines from d to any point on the part ad of the circumference of the parabola would be between the conic and its tangent, which is impossible. So the parabola must intersect the hyperbola at d and at another point, say a, and that is the required result.
This was the mistake that the honorable man made when he said that the two conics must be tangent to each other at \(d\).

His claim: “If \(bc\) was larger than \(ac\), then the problem has no solution because the two conic sections would not meet”, is a false claim. They could meet at one or two points or be tangent to each other between \(a\) and \(d\) as we have shown earlier. There is a more general proof than the one we presented:

Let the number of the squares be \(ab\) and \(bc\) be the side of the cube, which is more than half of \(ab\). Complete \(ch\) and then construct the two conic sections as we did before. Let \(ab\) be ten and \(zb\) be six, so the product of its square by \(az\) would be one hundred and forty-four, which equals the number and its side is \(bc\). There is no doubt that \(bc\) must be greater than five, because the cube of five is one hundred and twenty-five, so the parallelepiped whose base is the square of \(zb\) and height \(za\), equals the cube \(bc\). Consequently, their bases are equivalent to their heights; I mean the ratio of the square of \(zb\) to the square of \(bc\) is the same as the ratio of \(bc\) to \(za\). From \(z\) we draw a perpendicular which intersects the hyperbola at \(m\) and completes \(mb\). The rectangle \(mb\) is equal to the rectangle \(ch\) because their sides are equivalent; I mean the ratio of \(zb\) to \(bc\) is the same as the ratio of \(bc\) to \(zm\). Hence the ratio of the square of \(zb\) to the square of \(bc\) is the same as the ratio of \(zb\) to \(zm\), and this ratio is the same as the ratio of \(bc\) to \(za\). Thus the ratio of \(zb\) to \(zm\) is as the ratio of \(bc\) to \(za\), and by interchanging, the ratio of \(zb\) to \(bc\) is the same as \(zm\) to \(za\).

So the four lines are in sequence: \(zb\), \(bc\), \(zm\), \(za\). Thus the square of \(zm\) is equal to the product of \(bc\) and \(za\); and \(bc\) is the right side of the parabola whose axis is \(ab\) and vertex \(a\). It follows that \(zm\) is one of the lines of order.

So the point \(m\) is on the parabola. But it was on the hyperbola. Hence the two sections must meet. This shows that Abu-Aljood was wrong when he claimed that the sections would not meet. That is the required result.

To make it more clear, we let \(ab\) equal eighty and \(bc\) – which is the side of the cube that is equal to the number – equal forty-one, which is larger than \(ac\). So the point \(d\) lies outside the parabola. Assume the parabola passes through \(l\). Hence, the line \(lc\) is the root of one thousand five hundred and ninety-nine, which is slightly less than forty. We make \(tc\) equal \(bc\) and \(bm\) equal \(bt\); then we connect \(th\), so it is tangent to the parabola as we have shown. We take out \(ak\) to equal one fourth of \(ac\) and we draw a perpendicular to it that meets the conic section at \(p\). So the ratio of the square of \(lc\) to the square of \(kp\) is the same as the ratio of \(ac\) to \(ak\), because both are among the lines of order of the parabola.

Apollonius showed this in diagram \(yt\) of article \(a\), so \(kp\) would be half \(lc\) which is slightly less than twenty, and \(ct\) forty-one, and \(ak\) nine and three-fourths and \(at\) equals two. So \(tk\) is eleven and three-fourths, because the ratio of \(kz\) to \(kt\) is the same as the ratio of \(mb\) to \(bt\) and they are equal, so the line \(kz\) equals \(kt\), and hence the line \(zp\) is greater than eight, and it is inside the tangent line to the hyperbola, so in such a case, it must be within the hyperbola.

It is true that the two sections could not intersect if \(bc\) is larger than \(ca\), but that is not true in all cases, and Abu-Aljood made a mistake in his reasoning.
If you want to find numerical examples, you could find some as follows: “add a (volume of) parallelepiped to a given line, which is less than it by a cube, but equal to another given parallelepiped”.

If the side of the cube that is equal to the given parallelepiped equals or is less than half of the line, then it can be solved. If it is greater than half of the line, there could be cases with no possible solution, as we have shown.

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**THIS IS AN ARTICLE OF ABI-AL-FATH OMER BIN IBRAHEEM AL-KHAYAMI**

**In The Name of God, Most Gracious Most Merciful**

**From Him We Seek Exact Knowledge,**

**And on Him We Depend**

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We want to divide the quarter \( ab \) (arc) of the circle \( abcd \) into two halves, at the point \( z \), then draw the perpendicular \( zm \) to the diagonal \( bd \) so that the ratio of \( ah \) to \( zm \) is as the ratio of \( hm \) to \( mb \), where \( h \) is the center of the circle and \( ah \) is its radius.

We draw a circle with center \( h \), and draw a diagonal \( ac \). Then we draw another diagonal \( bd \) so that the two diagonals are orthogonal. From a point \( z \) (on the arc \( ab \)) we draw the perpendicular \( zm \) to \( bd \), such that the ratio of \( ah \) to \( zm \) is as the ratio of \( hm \) to \( mb \). Then, we draw the two perpendiculars \( kzt \) and \( tbn \), in such a way that \( bn \) equals (in length) \( ah \) and we complete the rectangle \( tl \).
Because the ratio of $ah$ to $zm$ is as the ratio of $hm$ to $mb$, and $bn$ equals $ah$, we get the ratio of $bn$ to $zm$ is the same as the ratio of $hm$ to $mb$. Further, the product of $bn$ and $mb$ equals the product of $zm$ and $hm$, as shown by Euclid in section $yo$ of article $o$ of the book of $Elements$. Further, the product of $bn$ and $mb$ equals the area of the rectangle $bl$, and the product of $zm$ and $hm$ equals the area of the rectangle $mk$, so the two rectangles $mk$ and $bl$ have the same area. We make the rectangle $mt$ common so the rectangle $th$ equals (in area) the rectangle $tl$.

Now, if we construct a hyperbola that does not meet the lines $kt$ and $tn$ but passes through the point $h$ (as shown by Apollonius in figure $nt$ of the first article of the book of conics, and in figures $o$ and $h$ of the second article of the same book), then this hyperbola must pass through the point $l$, as shown by reflecting figure eight of the second article of the book of sections.

The point $h$ is of known position, and the line $bn$ is of known position and magnitude. However, the point $l$ is not of known position when the section is
constructed, for if it were of known position, then the point \( m \) would be of known position because the line \( ml \) is of known magnitude. So the line \( bm \) would be of known magnitude, and consequently the whole figure would be totally known.

Also, the line \( tk \) is not of known position, for if it were, then the point \( t \) would be of known position. But if \( t \) were of known position, then the line \( tb \) would be of known magnitude. However, if \( tb \) were of known magnitude, then the whole figure would be totally known, which is not the case, since the goal is to determine the figure.

So if the point \( l \) were of known position, or the line \( tk \) were of known position, then the figure could be constructed and the conclusion would be reached easily. But knowing the location of either of them is not an easy task.

So if a researcher avoids this method and uses the book of sections, he will get the result in a different way.

I mentioned this method, though it is difficult, as a preliminary introduction for the student. I did not complete and build it geometrically because of its difficulty and the lack of many basics and introductions on conics. Whoever wants to continue this method using conic sections can do so, once he understands the method that I will present. Though this method also lacks some introductions on conic sections, it is much easier than the above-mentioned one, and more fruitful.

I say with the help of God:

From the construction we did at the beginning, we found the ratio of \( ah \) to \( zm \) equals the ratio of \( hm \) to \( mb \). From the point \( z \) we draw the line \( zt \) that is tangent to the circle, as Euclid explained in figure \( xo \) of section \( c \) of the book of Elements. We extend \( hb \) to intersect the tangent at \( t \) then we draw \( zh \). Since the triangle \( hzt \) is a right-angled triangle (at \( z \)), and the perpendicular \( zm \) was drawn from \( z \) to the base of the triangle, it follows from figure \( m \) of section \( y \) (of the book of Elements) that the ratio of \( hm \) to \( mz \) is as the ratio of \( mz \) to \( mt \). So the square of \( mz \) equals the product of \( hm \) and \( mt \). But the square of \( mz \) equals the product of \( dm \) and \( mb \). It follows that the product of \( dm \) and \( mb \) equals the product of \( hm \) and \( mt \), and consequently, the ratio of \( md \) to \( hm \) is as the ratio of \( mt \) to \( mb \), as shown in figure \( yo \) of section \( o \) (of the book of Elements). So the ratio of \( hc \) to \( hm \) is as the ratio of \( bt \) to \( mb \). But the ratio of \( ah \) to \( zm \) is as the ratio of \( hm \) to \( mb \) and by transitivity, we have the ratio of \( ah \) to \( hm \) is as the ratio of \( zm \) to \( mb \).

But it was shown that the ratio of \( ch \) to \( hm \) equals the ratio of \( bt \) to \( bm \). Hence the ratio of \( zm \) to \( mb \) is as the ratio of \( bt \) to \( mb \). Now, it is known (figure \( t \) of article \( h \) of the book of Elements) that quantities with the same ratio to a fixed quantity must be equal. Hence \( zm \) equals \( bt \), and \( zh \) equals \( hb \), and consequently \( hz \) plus \( zm \) equals \( bt \).

The analysis leads to a right-angled triangle, with the condition that the hypotenuse equals one of the two sides of the right angle plus the perpendicular from the right angle to the hypotenuse. Every time we construct such a right-angled triangle, we are able to put together this figure in the right geometrical way. This introduction, I mean this triangle with this property, is of great benefit.
in figures of these types. It has many other properties; we will mention some so the researcher can use it in many similar problems.

I say: this triangle cannot be an equilateral triangle. For, if the side $hz$ equals $zt$, then $hm$ equals $mt$, and the perpendicular would be equal to each one of them, and $ht$ would be equal to twice the perpendicular. Further, the sum of $hz$ and the perpendicular, which we assumed to equal the hypotenuse, would be greater than the hypotenuse. This is a contradiction.

And I say: $hz$ is smaller than $zt$, for if it were greater than it, then $hm$ would be greater than $mt$. Further, $mz$ (which is a middle line between the two lines $hm$ and $mt$) would be greater than $mt$. But it was assumed that $mz$ equals $tb$. So $tb$ would be greater than $tm$, which is impossible, since $tb$ is a piece of $tm$. Hence, the triangle as such, has the property that the side that is smaller than the perpendicular equals the one which is larger, and that is what we wanted to show.

Among other properties of such a triangle: The greater of the two sides of the right angle equals the sum of the smaller side plus the segment of the hypotenuse (formed by the perpendicular from the right angle to the hypotenuse) that meets the smaller side of the angle.

Our example will be based on the figure on page 48. I say: the sum (of the lengths) of $hz$ and $hm$ equals the side $zt$.

**Proof:** The ratio of $hd$ to $hm$ is as the ratio of $tb$ to $hm$, so by composition, we get the ratio of $dm$ to $mh$ is as the ratio of $tm$ to $mb$. Hence, the ratio of $dm$ to $mt$ is as the ratio $hm$ to $mb$. But the ratio of $hm$ to $mb$ is as the ratio of $hz$ to $zm$. And the ratio of $hz$ to $zm$ is as the ratio of $zt$ to $mt$ due to the similarity of the two triangles $hzt$ and $zmt$. Hence the ratio of $zt$ to $mt$ is as the ratio of $dm$ to $mt$, and this implies that $zt$ equals $md$, which in turn equals the sum of $hz$ and $hm$. So the sum of $hz$ and $hm$ equals $zt$, and that is what we wanted to show.
After what we have already introduced, we construct the triangle $abc$ with right angle at $b$. A perpendicular $bd$ is drawn from $b$ to $ac$. We assume that the sum of $ab$ and the perpendicular $bd$ equals $ac$, so the analysis leads to a known situation. Then we compose things to obtain the triangle with the above-mentioned property.

To follow in the steps of the honorable ancients, who were experts in simplifying abstract methods by using algebraists’ terminologies in these problems, we will use their terminology. These terminologies make multiplication and division much easier.

We draw $ad$ to be a specified length, say of length ten. Put $bd$ to be unknown, then multiply it by itself to make it square. Multiply the ten by itself to get one hundred, then sum to get a hundred and a square. This is the square of $ab$, as noted in figure $mz$ of article $a$ of the Elements. Further, the ratio of $ac$ to $ab$ is as the ratio of $ab$ to $ad$ because the two triangles $abc$ and $abd$ are similar. It follows that the product of $ac$ by $ad$ equals the square of $ab$. If we divide the square of $ab$ (which equals one hundred and a square) by $ad$, which equals ten, the outcome will be ten plus one tenth of a square, which equals $ac$. But we have assumed that $ac$ equals the sum of $ab$ and $bd$. Hence the sum of $ab$ and $bd$ equals ten and one tenth of a square. Subtracting $bd$ which is the unknown, from that, the result is ten plus one tenth of a square minus unknown (which is $ab$). Multiplying this by itself gives a hundred plus three squares plus one tenth of a tenth of a square minus twenty of the unknown and minus one fifth of cube equals a hundred plus square ($100 + 3x^2 + \frac{1}{100}x^3 - 20x - \frac{1}{5}x^3 = 100 + x^2$).

Simplifying gives two squares plus one tenth of tenth of a square-square equals twenty of the unknown plus one fifth of cube. Divide by the unknown to get twice the unknown plus one tenth of tenth of cube equals twenty plus one fifth of square. Multiply (both sides of the equation) by a hundred to get: cube plus two hundreds of the unknown equals twenty squares plus two thousands. This analysis lead to an equation involving four objects of different
ranks which cannot be solved using plane geometry of the cube, so conic sections will be needed.

Before we prove our claim using conics, we call the attention of the reader to an idea that will be an instigation to the reader of this article to acquire knowledge and master the part that we call attention to. Thanks to almighty God for his beneficences upon many of his worshippers. For, speaking of beneficences is great thanks to the giver, as God says in his book “speak of your God’s beneficences”. So the reader of this article should not think that this is showing off. Showing off is a practice of the arrogant, and they love it. Such people try to understand a little science. Once they have comprehended it (the little science), they think they know about all sciences, and God forbid that we allow ourselves to believe in such a way that prevents us from understanding the facts and winning salvation.

I say: what algebraists call square-square is an imaginary concept in continuous quantities. It has no existence in any way in materialistic objects. For continuous quantities, the terms square-square, square-cube and cube-cube are used to denote the number (coefficient) of the object (variable). Things share the sort of the quantity (all variables have the same type of coefficient: a real number), as is continuously shown by He who has the ultimate knowledge (God).

The things that algebraists use to denote objects and quantities are: number, root, square and cube. The number has to be taken as an abstract concept. It has no existence unless it is individuated by things. The root’s status in continuous quantities is as the status of the straight line (in geometrical figures). The square’s status is as that of a rectangle with equal sides and right angles whose side is what we call the root. The cube is the solid bounded by six equal squares whose sides are equal with right angles, any of its side is what we call the root and any of its faces is what is called the square, so the cube is the result of multiplying the root by itself, then multiplying the result by the root. This was shown and proved by Euclid in figure yz of article yc in the book of Elements.

Square-square, which, to the algebraists, is the product of the square by itself, has no meaning in continuous objects. This is because how can one multiply a square, which is a surface, by itself? Since the square is a two-dimensional object (geometrical figure), and two-dimensional by two-dimensional is a four-dimensional object. But solids cannot have more than three dimensions.

All objects in algebra are generated from these four genera. And anyone who says that algebra is a trick to determine unknown numbers is wrong. So don’t pay attention to these people. It is true that algebra and equations are geometrical things, as proved in article b of the book of Elements in figures h and w.

Now, whoever said: square-square plus three squares equals twenty-eight; he halved the squares then multiplied it by itself and then added the number; and took the root of the result to equal five and a half; then subtracted half the squares to get four which the square, and the square of the square is sixteen; and he thought that he deduced the square of the square using algebra: is very feeble in his thinking. This is because he did not deduce the square of the square but rather he deduced the square. It is exactly as if he said: square plus three roots
equals twenty-eight, then he determined the root using the second reduction, and concluded that the square of this root is the square of the square, which is a secret from which you will come to know other secrets.

Let us go back to what we were occupied with:

We say: the first three genera (sorts), I mean numbers, roots and squares, when equated come down to six different equations (branches): three of which are single-term equations, and three are multi-term equations. The unknowns can be determined using the second article as was mentioned and explained in the books of algebras. But if one studies equations involving cubes, then solids are needed especially cones and their sections, since the cube is a solid.

There are three single-term equations involving cubes: cube equals squares, which is equivalent to root equals numbers; and cube equals roots which is equivalent to square equals numbers, and cube equals numbers. There is no way to solve these equations except by using numerical methods intended to determine cubes; or else by using geometrical methods where a parallelepiped is constructed to equal a given parallelepiped. In such methods, conic sections are very much needed, and those who do not know conics need other techniques.

The multi-term equations involving cubes are of two types: either three-term equations or four-term equations. The three-term equations are:

(1) Cube plus squares equals numbers, which can be solved only by using conics.
(2) Cube plus squares equals roots, which can be treated as the equation: square plus roots equals numbers.
(3) Cube plus numbers equals roots, which can be solved only by using conics.
(4) Cube plus numbers equals squares, which can be solved only by using conics.
(5) Cube plus roots equals numbers, which can be solved only by using conics.
(6) Cube plus roots equals squares, which can be treated as square plus number equals root.
(7) Squares plus roots equals cube, which can be treated as roots plus numbers equals square.
(8) Squares plus numbers equals cube, which can be solved only by using conics.
(9) Roots plus numbers equals cube, which can be solved only by using conics.

So these are nine types of three-term equations, three of which can be deduced from the second book of *Elements*, and six of which can be solved only by using conics.

The four-term equations are:

(1) Cube equals squares plus roots plus numbers.
(2) Cube plus roots plus numbers equals squares.
(3) Cube plus squares plus numbers equals roots.
(4) Cube plus squares plus roots equals numbers.
(5) Cube plus square equals roots plus numbers.
(6) Cube plus roots equals squares plus numbers.
(7) Cube plus numbers equals squares plus roots.

These are seven types of four-term equations, none of which can be solved except by using conics.

So, there are thirteen multi-term equations involving cubes, all of which cannot be solved except by using conics; and one single-term equation that can be solved only by using conics, which is: cube equals numbers.

Ancient mathematicians, who did not speak our language, either did not do any work in relation to this, or their work did not reach us and was not translated to our language.

As for modern mathematicians who speak our language, the first one who needed a type of three-term equation out of the fourteen types, was Al-Mahani, the geometer. He was trying to prove the premise, which was taken for granted by Archimedes, in figure d of article b in the book on “the sphere and the cylinder”. Archimedes said: the two lines ab and bc are of known magnitude (length), joined along the same line. Further, the ratio of bc to ch is known, so ch is known, as shown in Data. Then he said: we make the ratio of cd to ch equal to the ratio of the square of ab to ad.

But he did not say how we know this, because necessarily this needs conic sections, and he did not introduce in the book (his work) anything that needs conics except this, and so he took it for granted.

And the fourth proposition was to try to divide the sphere by a plane into two parts in a pre-given ratio. Mahani used algebraists’ terminology to simplify things. But when analysis led him to equations involving numbers, squares and cubes, and he was not able to solve them by using conic sections, he concluded firmly that these equations have no solution. This gentleman, despite his deep understanding (and high achievement) of this industry (mathematics) could not solve certain of the previous types. Some time later, Abu Zafar Al-Khazin excelled in mathematics and discovered a new technique for solving these equations and explained it in an article (treatise). Also Abu Nasr Bin Iraq, a freedman of the
commander of the faithful, from the city of Khawarizm, was trying to solve the premise that was taken for granted by Archimedes to deduce the length of the side of a heptagon inscribed in a circle, which is based on the square having the above-mentioned property. He was using the terminology of the algebraists, and the analysis led him to: cube plus squares equals numbers, which he solved using conic sections. This man was certainly among the high-ranking mathematicians.

The problem that baffled Abu Sahel Al-Qohi, Aba Al-Wafaa Al-Bozjani, Aba Hamid Al-Saghani, and a group of their colleagues who devoted themselves to the Right Honorable “Adot Aldawla” (The governor) in the city of peace is: divide ten into two numbers so that the sum of their squares plus the outcome of dividing the greater by the smaller equals seventy-two. The analysis always led to squares equal cube plus roots plus numbers. These gentlemen were puzzled by the problem for a long time, till Abu-Aljood solved it and they kept the solution in the library of the Samanid kings. So these are three sorts: two of which are three-term equations, the third one is a four-term equation, while the fourth one is a single-term equation, I mean cube equals numbers, which has been solved by gentlemen who preceded us, but none of their work on the remaining ten sorts had reached us, nor any of their work on the above classification.

If I live for some time and success is my companion, I will write a self-contained article that includes the fourteen sorts with all of their types and branches, distinguishing the possible from the impossible ones – for several of the types require some conditions to be valid – and that contains many introductions that will be of great benefit to the principles of this industry, holding on the robe of success granted by God, depending on Him, for all seek help from him. He is the owner of power, praise be to His greatness.

Now, after these premises we go back to our problem, which is: determine a cube that satisfies the equation: the cube plus two hundreds of its side equals twenty squares of its side plus two thousands. To do that, we draw the line $ab$ to equal the number of the squares (which is twenty). We draw another line, $hz$, to equal two hundreds, and the line $hm$ to equal one, so the area of the rectangle $mz$ equals two hundreds. Then we draw a square, with side $am$, to equal the rectangle $mz$, as was deduced from figure $yd$ of article $b$ (of the book of Elements). Let the side $am$ be perpendicular to $ab$ which equals the root of two hundreds. Further, $ad$ equals ten, which is the result of dividing the number by the number of the roots; noting that the number is two thousands and the number of the roots is two hundreds. So dividing two thousands by two hundreds we get ten. Also $db$ equals ten.

On $db$ we construct the semicircle $dkh$, and we extend $dh$ to be parallel to $am$, then we complete the rectangle $ah$. Now, we draw the hyperbola $ndk$ that passes through $d$ and does not meet $am$ and $mh$, as was shown by the honorable Apollonius in proposition $nt$ of the first article of the book of conics, and in proposition $h$ and $w$ in the second article of that book. This construction cannot be done without these three propositions, which is: the hyperbola $ndk$ intersects the circle at $k$. From $k$ we construct $kl$ perpendicular to $ab$. I say: the side $al$ is
the side of the cube that satisfies: the cube plus two hundreds of its square equals twenty of the squares of al plus two thousands.

Proof: We extend lk until it intersects the line mh at t, then we draw kn parallel to al. Because kt is parallel to dh and kn is parallel to ad, the rectangle al equals the rectangle km. This is because the two points k and d are on a hyperbola that does not intersect the lines am and mt; and from each of the two points we draw two lines to the two other lines that do not meet the hyperbola and are parallel to their homologues (the other two lines from the other point). The honorable Apollonius proved this in figure c of article b of the book of conics. The circle dkb is of known location since its diagonal db is of known location and magnitude. The two lines am and mt are of known location, and the point p is of known location, so the conic ndk is of known location. Further, the point k and the line kl are of known locations. It follows that the point l is of known location. Also, the point a is of known location and consequently the line al is of known magnitude. All these deductions are clear in the book of Data.
We have shown that the rectangle \( ah \) equals the rectangle \( km \). Remove \( hp \), which is common, to get the rectangle \( dp \) equals \( kh \). We construct the rectangle \( dk \) common, so the rectangle \( ak \) equals the rectangle \( dt \), which are of equal angles because their angles are right angles. Hence their sides are equivalent in ratio as was shown by Euclid in proposition \( yd \) of article \( o \) (of the book of Elements), so the ratio of \( al \) to \( lt \) is as the ratio of \( dl \) to \( lk \), and hence their squares are proportional. Thus the ratio of the square of \( al \) to the square of \( lt \) is as the ratio of the square of \( dl \) to the square of \( lk \). Also, the ratio of \( dl \) to \( lk \) is as the ratio of \( lk \) to \( lb \). Consequently, the ratio of the square of \( dl \) to the square of \( lk \) is as the ratio of \( dl \) to \( lb \). It follows that the ratio of the square of \( al \) to the square of \( lt \) is as the ratio of \( dl \) to \( lb \). Hence the product of the square of \( al \) and the line \( lb \) equals the product of the square of \( lt \) and the line \( dl \). Take the product of the square of \( lt \) and \( ad \) to be a common factor, so the product of the square of \( lt \) and \( al \) equals the product of the square of \( lt \) and \( ad \) and equals the product of the square of \( al \) and \( lb \). But the square of \( lt \) equals the number of the sides, I mean two hundreds, and \( al \) is the side of the cube. So, two hundreds of the side of the cube equal the product of the square of \( lt \) and \( ad \) and equal the product of the square of \( al \) and \( lb \). But as we have shown, the product of the square of \( lt \) and \( ad \) equals two thousands. It follows that two thousands plus the product of the square of \( al \) and \( lb \) equals two hundreds of the side of the cube.

We take the cube of \( al \), which equals the product of the square of \( al \) by \( al \), a common factor. So the cube of \( al \) plus two hundreds of the side of the cube equals two thousands plus the product of the square of \( al \) and \( al \) plus the product of the square of \( al \) and \( lb \). But the product of \( al \) and \( al \) plus the product of the square of \( al \) and \( lb \) equals the product of the square of \( al \) and \( ab \). However we have assumed \( ab \) to equal twenty. Hence the product of the square of \( al \) and \( ab \) equals twenty of the squares of \( al \). So the cube of \( al \) plus two hundreds of (the length of) \( al \) equals two thousands plus twenty squares of the side of the cube. This is what we wanted to show.

Now, after what we have shown, we construct the triangle \( abc \) with \( ad \) (rational) equal to ten. Then \( db \) is the line \( al \) which we have proved is of known magnitude. I don’t mean of known (magnitude) length, for there is a difference. I mean by saying of known magnitude what Euclid meant in his book of Data: a line to which an equal magnitude (of its length) can be found.

So putting things in order, we can assume \( ad \) equals ten. We draw \( bd \) perpendicular to \( ad \) and equal to the line \( al \) in the preceding figure. Then we join \( ab \) and draw from \( b \) the perpendicular \( bc \). Extend \( ad \) to intersect the perpendicular from \( b \) at the point \( c \). This triangle \( abc \) must be a right-angled triangle at \( b \). So \( ab \) plus \( bd \) equals the hypotenuse \( ac \), and \( ab \) plus \( ad \) equals \( bc \), and that is what we wanted to show.

Consider the circle \( abcd \) with \( ab \) one quarter of its circumference. Draw the two diameters \( ac \) and \( bd \) to intersect at a right angle, and assume \( h \) is the center of the circle. Remove \( ch \) from the line \( cd \) of the triangle \( abc \) in the preceding figure, so that \( ch \) equals the perpendicular \( bd \). Divide the radius of the circle
(which equals \( \text{hb} \) in the preceding figure), at a point \( \text{m} \), into two parts in such a way that the ratio between them is as the ratio of \( \text{ad} \) to \( \text{dh} \) of the triangle \( \text{abc} \), as was shown by Euclid in proposition \( h \) of article \( w \) of his book, the \textit{Elements}. We draw the perpendicular \( \text{mz} \), and then we join \( \text{hz} \). From \( z \) we draw the tangent \( \text{zt} \) to the circle, and we extend \( \text{hb} \) until it intersects \( \text{zt} \) at the point \( t \). Hence the triangle \( \text{hzt} \) is similar to the triangle \( \text{abc} \) in the preceding figure.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (5,0) -- (3,3) -- (0,0);
\draw (0,0) node[below] {c} -- (5,0) node[below] {d} -- (3,3) node[above] {b} -- (0,0);
\draw (2.5,0) node {h};
\draw (1.5,2.5) node {a};
\draw (5,0) node {d};
\draw (3,3) node {b};
\draw (0,0) node {c};
\end{tikzpicture}
\end{center}

\textbf{Proof:} The angle \( \text{zhm} \) equals the angle \( \text{bac} \), for if not then one of them is larger than the other, say \( \text{bac} \). Draw from \( \text{h} \) on the line \( \text{hb} \) the angle \( \text{khl} \) that is equal to \( \text{bac} \). From \( \text{k} \), we draw the tangent \( \text{kl} \) to the circle. This tangent intersects \( \text{ht} \) at \( \lambda \), so the triangle \( \text{hkl} \) is similar to the triangle \( \text{abc} \) since their angles are equal. We draw the perpendicular \( \text{kn} \) from \( \text{k} \) to \( \text{hb} \) so \( \text{hk} \) plus \( \text{kn} \) equals \( \text{hl} \). Since \( \text{hb} \) equals \( \text{hk} \), it follows that \( \text{bl} \) equals \( \text{kn} \), and the ratio of \( \text{ln} \) to \( \text{kn} \) is as the ratio of \( \text{cd} \) to \( \text{db} \). Consequently, the ratio of \( \text{nl} \) to \( \text{lb} \) is as the ratio of \( \text{dc} \) to \( \text{ch} \). We conclude that the ratio of \( \text{nb} \) to \( \text{bl} \) is as the ratio of \( \text{dh} \) to \( \text{hc} \), and the ratio of \( \text{hc} \) (which is equal to \( \text{db} \)) to \( \text{da} \) is as the ratio of \( \text{bl} \) (which is equal to \( \text{kn} \)) to \( \text{nc} \). So from equal ratios we get the ratio of \( \text{hn} \) to \( \text{nb} \) is as the ratio of \( \text{ad} \) to \( \text{dh} \). But we have made the ratio of \( \text{hm} \) to \( \text{mb} \) equal to the ratio of \( \text{ad} \) to \( \text{dh} \), so the ratio of \( \text{hn} \) to \( \text{nb} \) is as the ratio of \( \text{hm} \) to \( \text{mb} \). However, \( \text{hn} \) (the first) is smaller than \( \text{hm} \) (the third). It follows necessarily that \( \text{nb} \) (the second) is smaller than \( \text{mb} \) (the fourth), based on proposition \( \text{vd} \) in chapter five of the book of \textit{Elements}. But it was assumed that it is larger. This is impossible. So the angle \( \text{zhm} \) is not smaller than \( \text{bac} \) in the triangle \( \text{abc} \), neither could it be larger than it; so the triangle \( \text{zhm} \) is similar to \( \text{abc} \). Consequently \( \text{hz} \) plus \( \text{zm} \) equals \( \text{ht} \). It follows that \( \text{bt} \) equals \( \text{zm} \), and the product of \( \text{dm} \) and \( \text{mb} \) equals the square of \( \text{mz} \). Also, the product of \( \text{hm} \) and \( \text{mt} \) equals the square of \( \text{mz} \). Hence the product of \( \text{dm} \) and \( \text{mb} \) equals the product of \( \text{hm} \) and \( \text{mt} \). So the four lines are proportional, as shown in proposition \( \text{vo} \) of article \( \text{d} \) (of the book of \textit{Elements}), and the ratio of \( \text{dm} \) (the first) to \( \text{mb} \) (the second) is as the ratio of \( \text{mt} \) (the third) to \( \text{mb} \) (the fourth). In conclusion: the ratio of \( \text{hd} \) to \( \text{hm} \) is as the ratio of \( \text{bt} \) to \( \text{bm} \), and \( \text{dh} \) equals \( \text{ah} \) and \( \text{bt} \) equals \( \text{zm} \). So the ratio of \( \text{ah} \) to \( \text{hm} \) is as the ratio of \( \text{zm} \) to \( \text{mb} \). By interchanging ratios we get the ratio of \( \text{ah} \) to \( \text{zm} \) is as the ratio of \( \text{hm} \) to \( \text{mb} \).
Hence we divided a quarter of the circumference of a circle into two pieces at a point $z$, from which we drew the perpendicular $zm$ so that the ratio of $ah$ (which is the radius) to $zm$ equals the ratio of $hm$ to $mb$, and that is what we wanted to show.

Whoever wants to deduce the above result using arithmetic will not be able to do so, since results that are deduced using conic sections cannot be deduced using arithmetic. But if he is satisfied by guessing (and approximating), then he must go back to the tables of hypotenuses in the book of Al-Magest, or the tables of sines and arrows of a reliable ephemeris. He must look for an arc (of the circle) where the ratio of sixty (which is the radius of the circle by assumption) to its sine (the central angle with this arc) is as the ratio of the cosine to its arrow. So one finds this arc to be very close to $57$ ($nz$) degrees of the $360$ ($shs$) degrees of the circle, its sine is close to $50$ ($n$) degrees, its arrow is close to $27$ ($kz$) parts and one third of a part, and its cosine is close to thirty-two degrees and two thirds of a degree. One can go on refining these estimates until no difference can be noticed between the exact and the approximate values.

This is what was possible to discuss and present in that direction, despite our thoughts being scattered, and despite the many worries of life that hinder us from tackling these particular cases. And had it not been for the nobility of the council (may his nobility be for ever) and the right of the querist (may God support him always), I would not bother with these small details, but rather occupy myself with what is more important. We thank God in all cases, and in God we trust to guide us to good things in life.

The article is complete, and peace be upon the Seal of the Prophecy.
Problem
Given a quarter of a circle $ac$ with assumed center $b$, and we want to divide $ab$ into two parts, as you already know.

We construct on $ab$ a square (equal sides and angles), say $ah$, so that the lines $bh$ and $hd$ are of known location, and the point $a$ is of known location. At $a$ we draw the hyperbola $az$ that does not meet the lines $bh$ and $hd$, so this hyperbola is of known location. We join $ac$, and necessarily it is tangent to the hyperbola and it is inside the circle. Consequently, the hyperbola must intersect the circle, say at $z$, and $z$ will be of known location. We draw the two perpendiculars $zm$ and $zk$. I say the work is done.

Proof: The two points $a$ and $z$ are on the hyperbola. From each of these two points, two lines were drawn to meet the two lines that do not meet the hyperbola and parallel to the two lines from the other point. So the rectangle $zh$ equals the rectangle $ah$. We remove $kh$, which is common to both. What is left is $km$, which equals $kd$, and the corresponding angles of the two rectangles are equal, so their sides are equivalent: the ratio of $ad$ to $kz$ is as the ratio of $bk$ to $ka$. 
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